ECE 486: Control Systems

Lecture 13A: Steady-State Sinusoidal Response

Key Takeaways

The transfer function *G(s)* is used to express the solution of a stable linear system forced by a sinusoidal input.

If the input is $u(t) = \sin(\omega t)$ then the response satisfies:

$$y(t) \to |G(j\omega)| \sin(\, \omega t + \angle G(j\omega)\,) \text{ as } t \to \infty$$

The output converges to a sinusoid at the same frequency as the input but with amplitude scaled by $|G(j\omega)|$ and phase is shifted by $\angle G(j\omega)$.

Revisiting The Transfer Function

The transfer function G(s) was introduced as notation for an ODE.

Now we'll think of it as a function that takes a complex number *s* as input and returns a complex number *G(s)*.

The response of the ODE to a sinusoidal input depends on the transfer function evaluated at a purely imaginary number $s = j\omega$ where $\omega > 0$ is the frequency in rad/sec .

The result *G(jw)* is a complex number that can be expressed in

- Cartesian form by its real and imaginary parts, or
- Polar form by its magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$.

Example

- Stable, first-order system:

$$\dot{y}(t) + 4y(t) = 2u(t) \qquad G(s) = \frac{2}{s+4}$$

- Evaluate at $\omega = 3\, rad/sec$

$$G(3j) = \frac{2}{3j+4}$$

- Cartesian Form:

$$G(3j) = \frac{2}{3j+4} \cdot \frac{4-3j}{4-3j} = \frac{8-6j}{25} = 0.32 - 0.24j$$

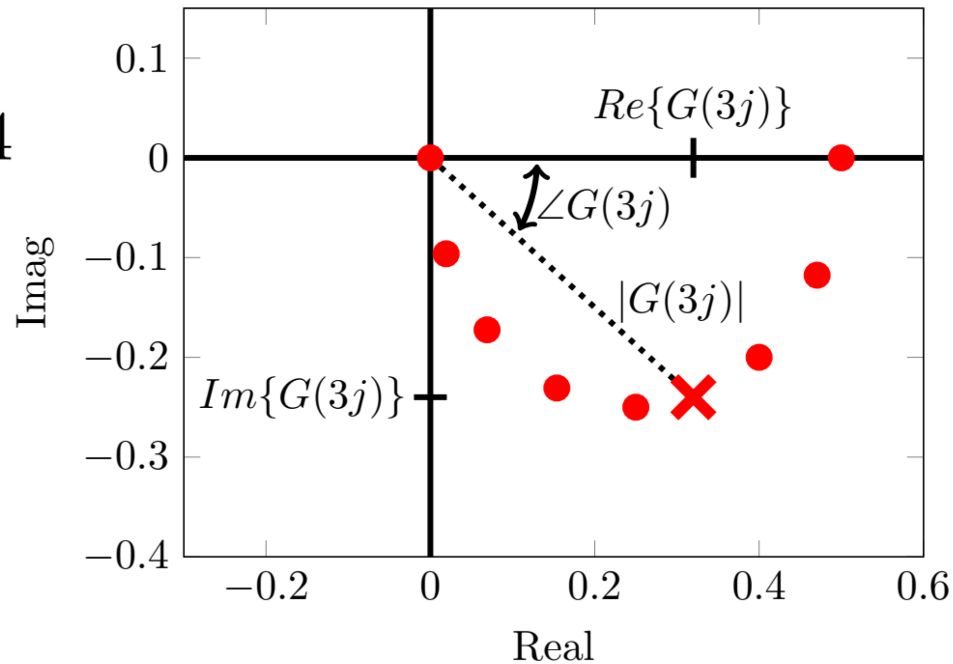
- Polar Form:

$$|G(3j)| = \sqrt{0.32^2 + 0.24^2} = 0.4$$

$$\angle G(3j) = \tan^{-1}\left(-\frac{0.24}{0.32}\right)$$

$$= -0.64 rads$$

$$G(j\omega) = 0.4e^{-0.64j}$$



Sinusoidal Response: First-Order Systems

Consider the stable, first-order system: $G(s) = \frac{b_0}{s+a_0}$

$$\dot{y}(t) + a_0 y(t) = b_0 u(t) \text{ with } y(0) = y_0$$

First consider complex exponential inputs: $u(t) = e^{j\omega t}$

The characteristic equation has one root: $s_1 = -a_0 < 0$

The general form of the forced-response solution is:

$$y(t) = y_P(t) + c_1 e^{-a_0 t}$$

"Guess" the particular solution: $y_P(t) = c_P e^{j\omega t}$

Sub into the ODE:

$$j\omega c_P\, e^{j\omega t} + a_0 c_P e^{j\omega t} = b_0 e^{j\omega t} \quad \Longrightarrow \quad c_P = \frac{b_0}{j\omega + a_0} = G(j\omega)$$

General solution:

$$y(t) = G(j\omega)e^{j\omega t} + c_1 e^{-a_0 t} \quad \Longrightarrow \quad y(t) \to G(j\omega)e^{j\omega t} \text{ as } t \to \infty$$

(Convergence depends on $\tau_1 = \frac{1}{a_0}$)

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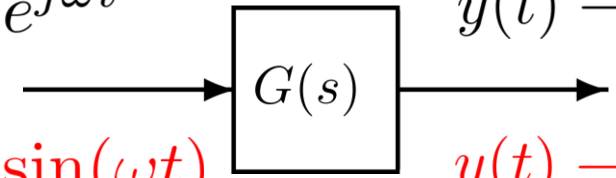
Transfer function in polar form: $G(j\omega) = |G(j\omega)|e^{j\angle G(j\omega)}$

Recall Euler's formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

Take imaginary part of complex solution

$u(t) = e^{j\omega t}$ $\quad$ $y(t) \to G(j\omega)e^{j\omega t} = |G(j\omega)|e^{j(\omega t + \angle G(j\omega))}$



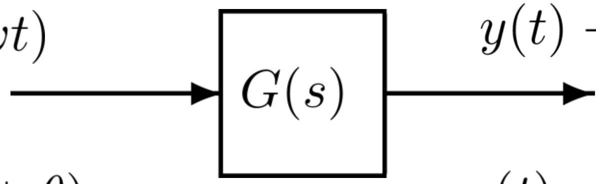
$u(t) = \sin(\omega t)$ $\quad$ $y(t) \to |G(j\omega)|\sin(\omega t + \angle G(j\omega))$

Steady-State Sinusoidal Response

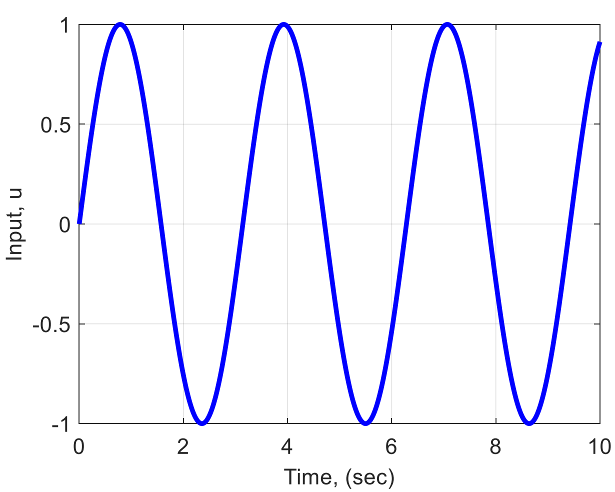
Consider a stable, n^{th}-order system with transfer function:

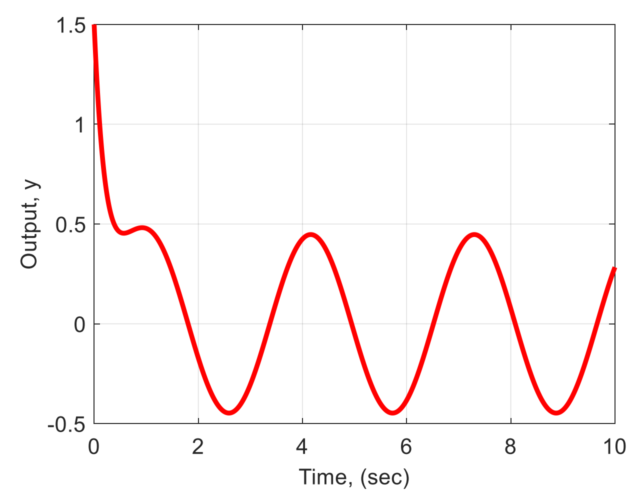
$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

$$u(t) = \sin(\omega t) \qquad y(t) \to |G(j\omega)| \sin\left(\omega t + \angle G(j\omega)\right)$$



$$u(t) = A \sin(\omega t + \theta) \qquad y(t) \to A|G(j\omega)| \sin\left(\omega t + \theta + \angle G(j\omega)\right)$$





Example

- Consider the following stable, first-order system:

$$\dot{y}(t) + 4y(t) = 2u(t) \text{ with IC: } y(0) = 1.5 \qquad G(s) = \frac{2}{s+4}$$

- Find response due to $u(t) = \sin(2t)$
- Evaluate transfer function:

$$\omega = 2\frac{rad}{sec} \quad \Rightarrow \quad |G(2j)| = 0.447 \text{ and } \angle G(2j) = -0.464 rads$$

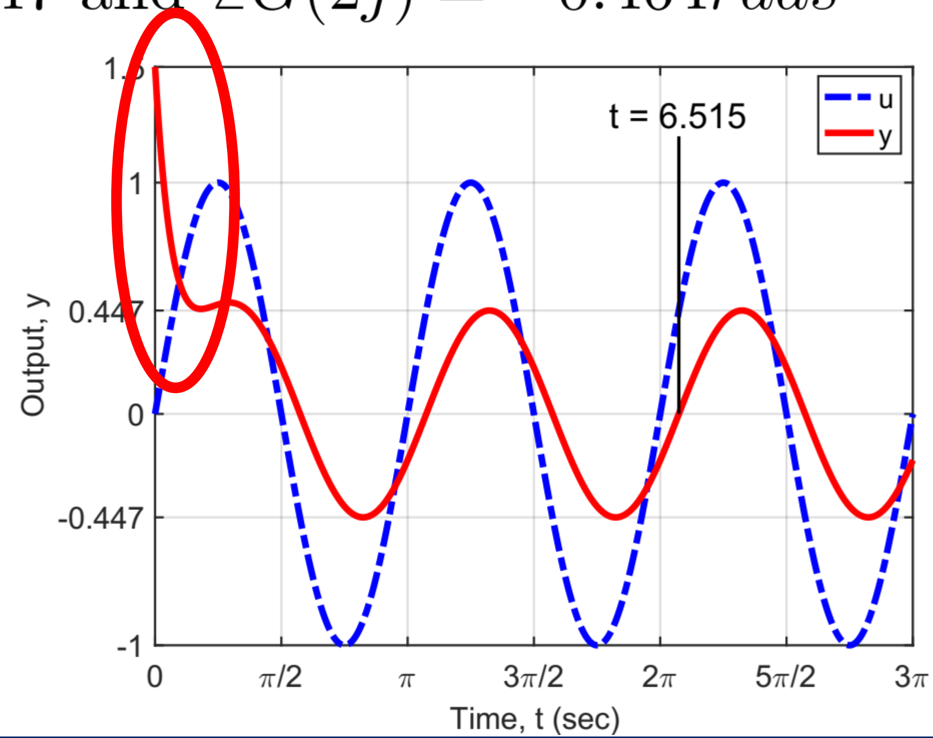
- Sinusoidal response:

$$y(t) \to 0.447 \sin(2t - 0.464)$$

Time constant is $\tau = \frac{1}{4} sec$

Transient decays after

$$3\tau = 0.75 sec$$



Leading vs. Lagging Response

- Steady-state sinusoidal response:

$$y(t) \to |G(j\omega)| \sin\left(\omega t + \angle G(j\omega)\right)$$

- Re-write as:

$$y(t) \to |G(j\omega)| \sin\left(\omega(t - t_{shift})\right) \text{ where } t_{shift} := -\frac{\angle G(j\omega)}{\omega}$$

- Terminology:

Lagging: $\angle G(j\omega) < 0 \Rightarrow t_{shift} > 0$
Leading: $\angle G(j\omega) > 0 \Rightarrow t_{shift} < 0$

$$y(t) \to 0.447 \sin(2t - 0.464)$$

$$y(t) \to 0.447 \sin(2(t - 0.232))$$

$u(2\pi) = 0$ and $y(t) = 0$ at
$t = 2\pi + 0.232 \approx 6.515 sec.$