Plan of the Lecture

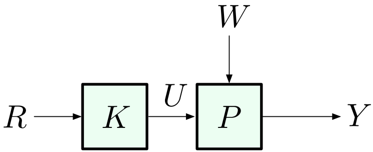
- Review: stability; Routh–Hurwitz criterion
- Today's topic: basic properties and benefits of feedback control

Goal: understand the difference between open-loop and closed-loop (feedback) control; examine the benefits of feedback: reference tracking and disturbance rejection; reduction of sensitivity to parameter variations; improvement of time response.

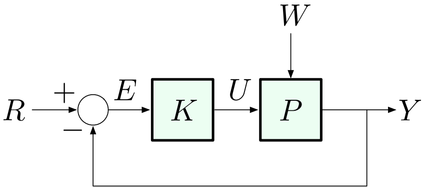
Reading: FPE, Section 4.1; lab manual

Two Basic Control Architectures

- Open-loop control

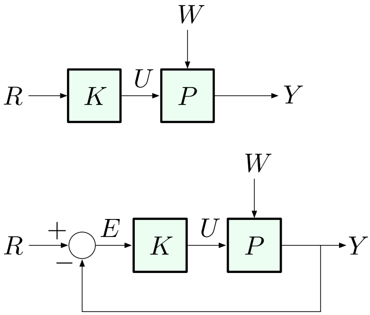


- Feedback (closed-loop) control



Here, W is a *disturbance*; K is *not necessarily* a static gain

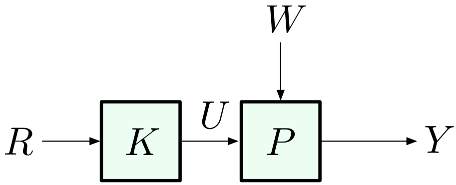
Basic Objectives of Control



- track a given reference
- reject disturbances
- meet performance specs

Intuitively, the difference between the open-loop and the closed-loop architectures is clear (think cruise control ...)

Open-Loop Control



- cheaper/easier to implement (no sensor required)
- does not destabilize the system

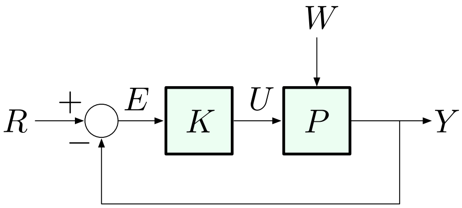
e.g., if both K and P are stable (all poles in OLHP),

$$\frac{Y}{R} = KP$$

is also stable:

$$\{\text{poles of } KP\} = \{\text{poles of } K\} \cup \{\text{poles of } P\}$$

Feedback Control

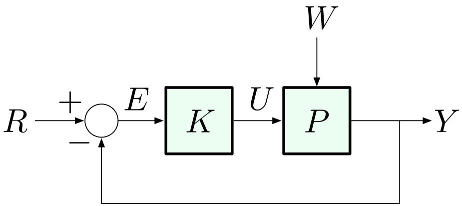


- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)
- may destabilize the system:

$$\frac{Y}{R} = \frac{KP}{1 + KP}$$

  has new poles, which may be unstable
- but: feedback control is the *only way* to stabilize an unstable plant (this was the Wright brothers' key insight)

Benefits of Feedback Control



Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Case Study: DC Motor

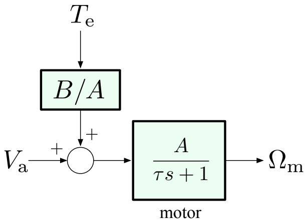
Inputs:	v_{a} – input voltage
	τ_{e} – load/disturbance torque
Outputs:	ω_{m} – angular speed of the motor

Transfer function:

$$\Omega_{\mathrm{m}} = \frac{A}{\tau s + 1} V_{\mathrm{a}} + \frac{B}{\tau s + 1} T_{\mathrm{e}}$$

τ – time constant

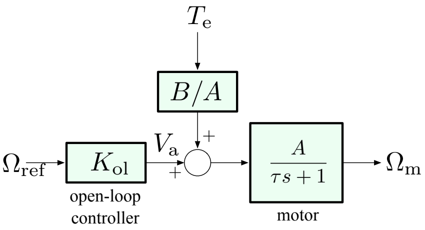
A, B – system gains



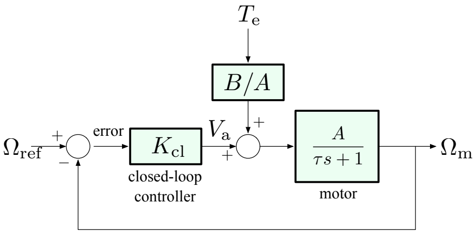
Objective: have Ω_{m} approach and track a given reference Ω_{ref} in spite of disturbance T_{e}.

Two Control Configurations

- Open-loop control



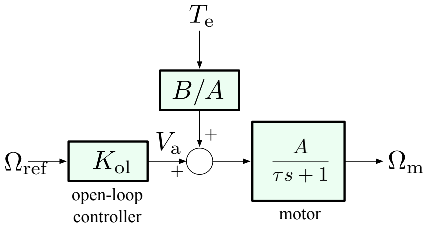
- Feedback (closed-loop) control



Disturbance Rejection

Goal: maintain $\omega_{\rm m} = \omega_{\rm ref}$ in steady state in the presence of *constant* disturbance.

Open-loop:

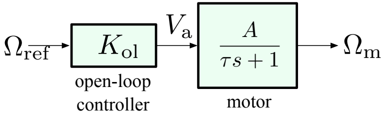


– the controller receives *no information* about the disturbance $\tau_{\rm e}$ (the only input is $\omega_{\rm ref}$, no feedback signal from anywhere else)

– so, let's attempt the following: design for *no disturbance* (i.e., $\tau_{\rm e} = 0$), then see how the system works in general

Disturbance Rejection: Open-Loop Control

First assume zero disturbance:



Transfer function:

$$\frac{A}{\tau s+1} \quad \text{(stable pole at } s=-1/\tau\text{)}$$

We want DC gain $= 1$

$$\Omega_{\mathrm{m}} = \frac{A}{\tau s+1} V_{\mathrm{a}} = \frac{K_{\mathrm{ol}} A}{\tau s+1} \Omega_{\mathrm{ref}}$$

Let's just use constant gain: $K_{\mathrm{ol}} = 1/A$

$$\omega_{\mathrm{m}}(\infty) = \frac{1}{A} \cdot A \cdot \omega_{\mathrm{ref}} = \omega_{\mathrm{ref}} \qquad (\text{for } T_{\mathrm{e}} = 0)$$

What happens in the presence of nonzero T_{e}?

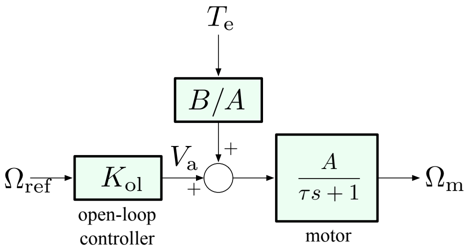
$$\Omega_{\mathrm{m}} = \underbrace{\frac{A}{\tau s+1} \frac{1}{A}}_{\text{DC gain}=1} \Omega_{\mathrm{ref}} + \underbrace{\frac{B}{\tau s+1}}_{\text{DC gain}=B} T_{\mathrm{e}}$$

$$\Longrightarrow \omega_{\mathrm{m}}(\infty) = \underbrace{\omega_{\mathrm{ref}}}_{\text{step input}} + B \underbrace{\tau_{\mathrm{e}}}_{\text{step input}}$$

Disturbance Rejection: Open-Loop Control

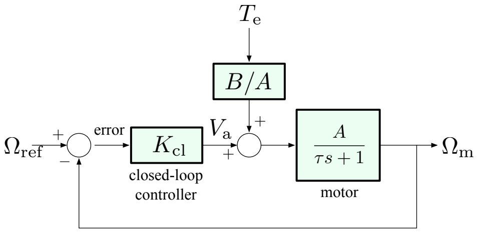
Steady-state motor speed for constant reference and constant disturbance:

$$\omega_{\mathrm{m}}(\infty) = \omega_{\mathrm{ref}} + B\tau_{\mathrm{e}}$$



Conclusion: in the absence of disturbances, reference tracking is good, but disturbance rejection is pretty poor. Steady-state error is determined by B, and we have no control over it (and, in fact, cannot change this through any choice of controller K_{ol}, no matter how clever)

Disturbance Rejection: Feedback Control



$$V_a = K_{cl}E = K_{cl}\left(\Omega_{ref} - \Omega_m\right)$$

$$\Omega_m = \frac{A}{\tau s+1}K_{cl}\left(\Omega_{ref} - \Omega_m\right) + \frac{B}{\tau s+1}T_e$$

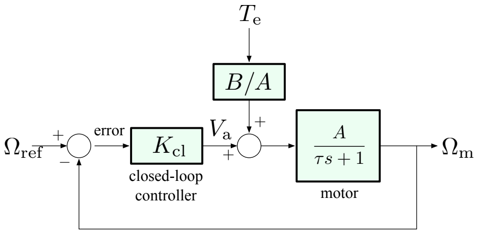
Solve for Ω_m:

$$(\tau s+1)\Omega_m = AK_{cl}\left(\Omega_{ref} - \Omega_m\right) + BT_e$$

$$(\tau s+1+AK_{cl})\Omega_m = AK_{cl}\Omega_{ref} + BT_e$$

$$\Omega_m = \frac{AK_{cl}}{\tau s+1+AK_{cl}}\Omega_{ref} + \frac{B}{\tau s+1+AK_{cl}}T_e$$

Disturbance Rejection: Feedback Control



$$\Omega_m = \underbrace{\frac{AK_{cl}}{\tau s + 1 + AK_{cl}}}_{\text{DC gain}=\frac{AK_{cl}}{1+AK_{cl}}} \Omega_{ref} + \underbrace{\frac{B}{\tau s + 1 + AK_{cl}}}_{\text{DC gain}=\frac{B}{1+AK_{cl}}} T_e$$

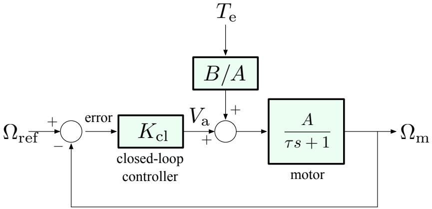
(provided all transfer functions are strictly stable)

Assuming that the reference ω_{ref} and the disturbance τ_e are constant, we apply FVT:

$$\omega_m(\infty) = \frac{AK_{cl}}{1 + AK_{cl}} \omega_{ref} + \frac{B}{1 + AK_{cl}} \tau_e$$

Disturbance Rejection: Feedback Control

Steady-state speed for constant reference and disturbance:



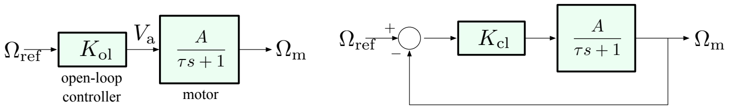
$$\omega_m(\infty) = \frac{AK_{cl}}{1+AK_{cl}}\omega_{ref} + \frac{B}{1+AK_{cl}}\tau_e$$

Conclusions:

- $\frac{AK_{cl}}{1+AK_{cl}} \neq 1$, but can be brought arbitrarily close to 1 when $K_{cl} \to \infty$. Thus, steady-state tracking is good with high gain, but never quite as good as in open-loop case.
- $\frac{B}{1+AK_{cl}}$ is small (arbitrarily close to 0) for large K_{cl}. Thus, *much* better disturbance rejection than with open-loop control.

Sensitivity to Parameter Variations

Consider again our DC motor model, with no disturbance:



Bode's sensitivity concept: In the "nominal" situation, we have the motor with DC gain $= A$, and the overall transfer function, either open- or closed-loop, has some other DC gain (call it T).

Now suppose that, due to modeling error, changes in operating conditions, etc., the motor gain changes:

$$A \longrightarrow A + \underbrace{\delta A}_{\substack{\text{small} \\ \text{perturbation}}}$$

This will cause a perturbation in the overall DC gain:

$$T \longrightarrow T + \delta T \qquad \text{(from calculus, to 1st order, } \delta T \approx \frac{\mathrm{d}T}{\mathrm{d}A}\delta A\text{)}$$

Sensitivity to Parameter Variations

$$A \longrightarrow A + \delta A \qquad \text{(small perturbation in system gain)}$$
$$T \longrightarrow T + \delta T \qquad \text{(resultant perturbation in overall DC gain)}$$

Hendrik Wade Bode
(1905–1982)

Bode's sensitivity:

$$\mathcal{S} \triangleq \frac{\delta T / T}{\delta A / A}$$

$$\begin{aligned} \mathcal{S} &= \text{relative error} \\ &= \frac{\text{normalized (percentage) error in } T}{\text{normalized (percentage) error in } A} \end{aligned}$$

Sensitivity to Parameter Variations

Let's compute $\mathcal{S}$ for our DC motor control example, both open- and closed-loop.

Open-loop:

- nominal case $\quad T_{\rm ol} = K_{\rm ol} A = \frac{1}{A} A = 1$
- perturbed case

$$A \longrightarrow A + \delta A$$

$$T_{\rm ol} \longrightarrow K_{\rm ol}(A + \delta A) = \underbrace{\frac{1}{A}}_{\substack{\text{design}\\ \text{choice}}} (A + \delta A) = \underbrace{1}_{T_{\rm ol}} + \underbrace{\frac{\delta A}{A}}_{\delta T_{\rm ol}}$$

Sensitivity: $\quad \mathcal{S}_{\rm ol} = \dfrac{\delta T_{\rm ol}/T_{\rm ol}}{\delta A_{\rm ol}/A_{\rm ol}} = \dfrac{\delta A/A}{\delta A/A} = 1$

For example, a 5% error in A will cause a 5% error in $T_{\rm ol}$.

Sensitivity to Parameter Variations

Closed-loop:

- nominal case $\qquad T_{\mathrm{cl}} = \dfrac{AK_{\mathrm{cl}}}{1 + AK_{\mathrm{cl}}}$
- perturbed case

$$A \longrightarrow A + \delta A \qquad T_{\mathrm{cl}} \longrightarrow T_{\mathrm{cl}} + \underbrace{\delta T_{\mathrm{cl}}}_{\substack{\text{how to} \\ \text{compute this?}}}$$

Taylor expansion:

$$T(A + \delta A) = T(A) + \frac{\mathrm{d}T}{\mathrm{d}A}(A)\delta A + \text{higher-order terms}$$

In our case:

$$\frac{\mathrm{d}T_{\mathrm{cl}}}{\mathrm{d}A} = \frac{K_{\mathrm{cl}}}{1 + AK_{\mathrm{cl}}} - \frac{AK_{\mathrm{cl}}^2}{(1 + AK_{\mathrm{cl}})^2} = \frac{K_{\mathrm{cl}}}{(1 + AK_{\mathrm{cl}})^2}$$

$$\delta T_{\mathrm{cl}} = \frac{K_{\mathrm{cl}}}{(1 + AK_{\mathrm{cl}})^2}\delta A$$

Sensitivity to Parameter Variations

From before:

$$\delta T_{\mathrm{cl}} = \frac{K_{\mathrm{cl}}}{(1 + AK_{\mathrm{cl}})^2} \delta A$$

$$T_{\mathrm{cl}} = \frac{AK_{\mathrm{cl}}}{1 + AK_{\mathrm{cl}}}$$

Therefore

$$\delta T_{\mathrm{cl}} / T_{\mathrm{cl}} = \frac{\frac{K_{\mathrm{cl}}}{(1+AK_{\mathrm{cl}})^2} \delta A}{\frac{AK_{\mathrm{cl}}}{1+AK_{\mathrm{cl}}}} = \frac{1}{1 + AK_{\mathrm{cl}}} \delta A / A$$

Sensitivity: $\mathcal{S}_{\mathrm{cl}} = \dfrac{\delta T_{\mathrm{cl}} / T_{\mathrm{cl}}}{\delta A / A} = \dfrac{1}{1 + AK_{\mathrm{cl}}}$ $\quad (\ll 1 \text{ for large } K_{\mathrm{cl}})$

With high-gain feedback, we get smaller relative error due to parameter variations in the plant model.

Time Response

We still assume no disturbance: $\tau_{\mathrm{e}} = 0$.

So far, we have focused on DC gain only (steady-state response). What about *transient response*?

Open-loop

$$\Omega_{\mathrm{m}} = \frac{AK_{\mathrm{ol}}}{\tau s + 1}\Omega_{\mathrm{ref}}$$

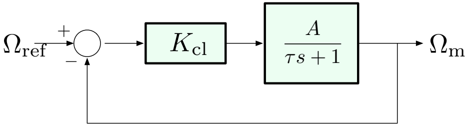
Pole at $s = -\frac{1}{\tau}$ $\quad\Longrightarrow\quad$ transient response is $e^{-t/\tau}$

Here, τ is the *time constant*: the time it takes the system response to decay to $1/e$ of its starting value.

In the open-loop case, smaller time constant means faster convergence to steady state. This is not affected by the choice of K_{ol} in any way!

Time Response

Closed-loop



$$\Omega_{\mathrm{m}} = \frac{AK_{\mathrm{cl}}}{\tau s + 1 + AK_{\mathrm{cl}}} \Omega_{\mathrm{ref}}$$

Closed-loop pole at $s = -\frac{1}{\tau}(1 + AK_{\mathrm{cl}})$
(the only way to move poles around is *via feedback*)

Now the transient response is $e^{-\frac{1+AK_{\mathrm{cl}}}{\tau}t}$, with

$$\text{time constant} = \frac{\tau}{1 + AK_{\mathrm{cl}}}$$

— for large K_{cl}, we have a much smaller time constant, i.e., *faster convergence* to steady-state.

Summary

Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Word of caution: high-gain feedback only works well for 1st-order systems; for higher-order systems, it will typically cause underdamping and instability.

Thus, we need a more sophisticated design than just static gain. We will take this up in the next lecture with *Proportional-Integral-Derivative* (PID) control.