CS440/ECE448 Lecture 8: Linear Classifiers

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Outline

- Linear Classifiers
- Gradient descent
- Cross-entropy
- Softmax

Linear classifier: Notation

- The observation $x = [x_0, \dots, x_{D-1}]$ is a real-valued vector (D is its dimension)
- The class label $y \in \mathcal{Y}$ is drawn from some finite set of class labels.
- Usually the output vocabulary, $\mathcal{Y}$, is some set of strings. For convenience, though, we usually map the class labels to a sequence of integers, $\mathcal{Y} = \{0, \dots, V-1\}$, where V is the vocabulary size

Linear classifier: Definition

A linear classifier is defined by

$$f(x) = \underset{k}{\operatorname{argmax}}\, w_k @ x + b_k$$

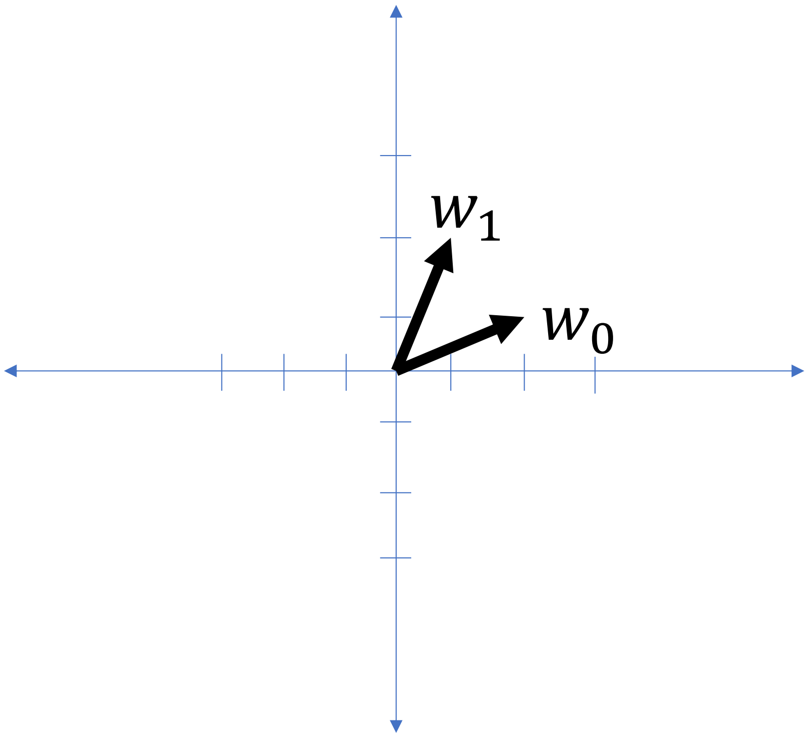
- @ means matrix product or dot product, $w_k @ x = \sum_{j=0}^{D-1} x_j w_{k,j}$
- w_k, b_k are the **weight vector** and **bias** corresponding to **class $\boldsymbol{k}$**.
- There are a total of $V(D+1)$ trainable parameters:

$$\begin{aligned}(\#\text{ params}) &= (\#\text{ classes}) \times \big(\text{len}(w_k) + \text{len}(b_k)\big) \\ &= V(D+1)\end{aligned}$$

Example

Consider a two-class classification problem, with the biases $b_0 = b_1 = 0$, and

$$w_0 = [2,1]$$
$$w_1 = [1,2]$$



Example

Notice that in the two-class case, the equation

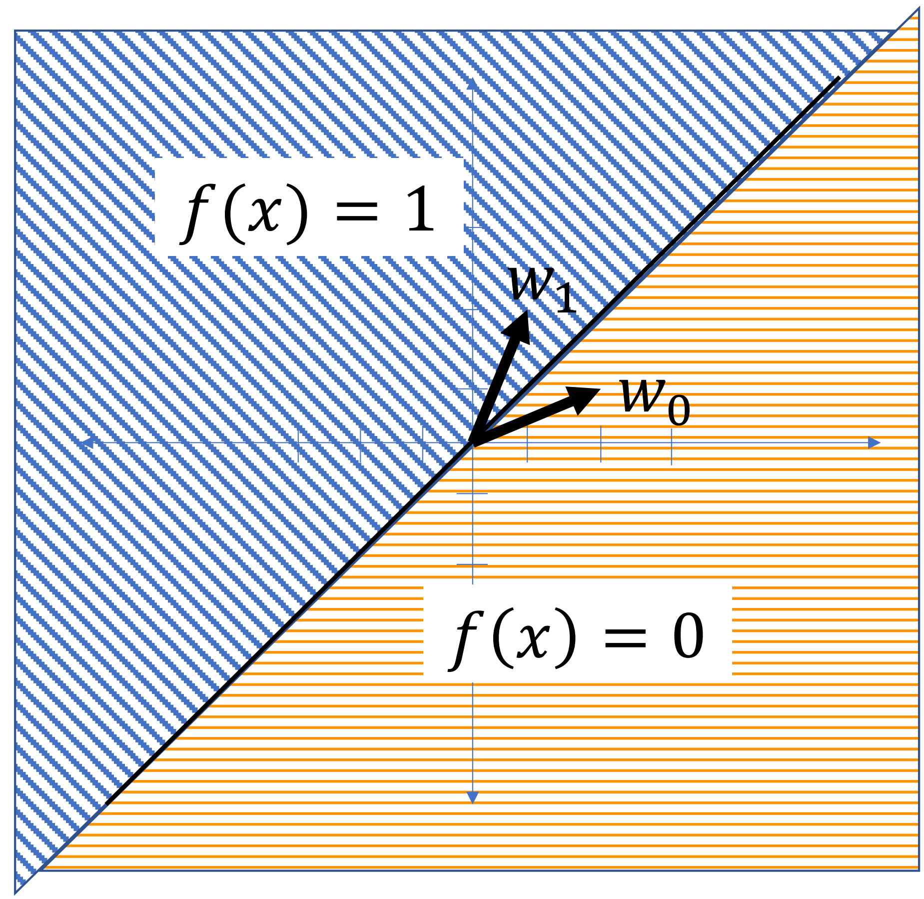
$$f(x) = \underset{k}{\operatorname{argmax}}\, w_k@x + b_k$$

Simplifies to

$$f(x) = \begin{cases} 1 & w_1@x + b_1 > w_0@x + b_0 \\ 0 & w_1@x + b_1 < w_0@x + b_0 \end{cases}$$

The class boundary is the line whose equation is

$$(w_1 - w_0)@x + (b_1 - b_0) = 0$$



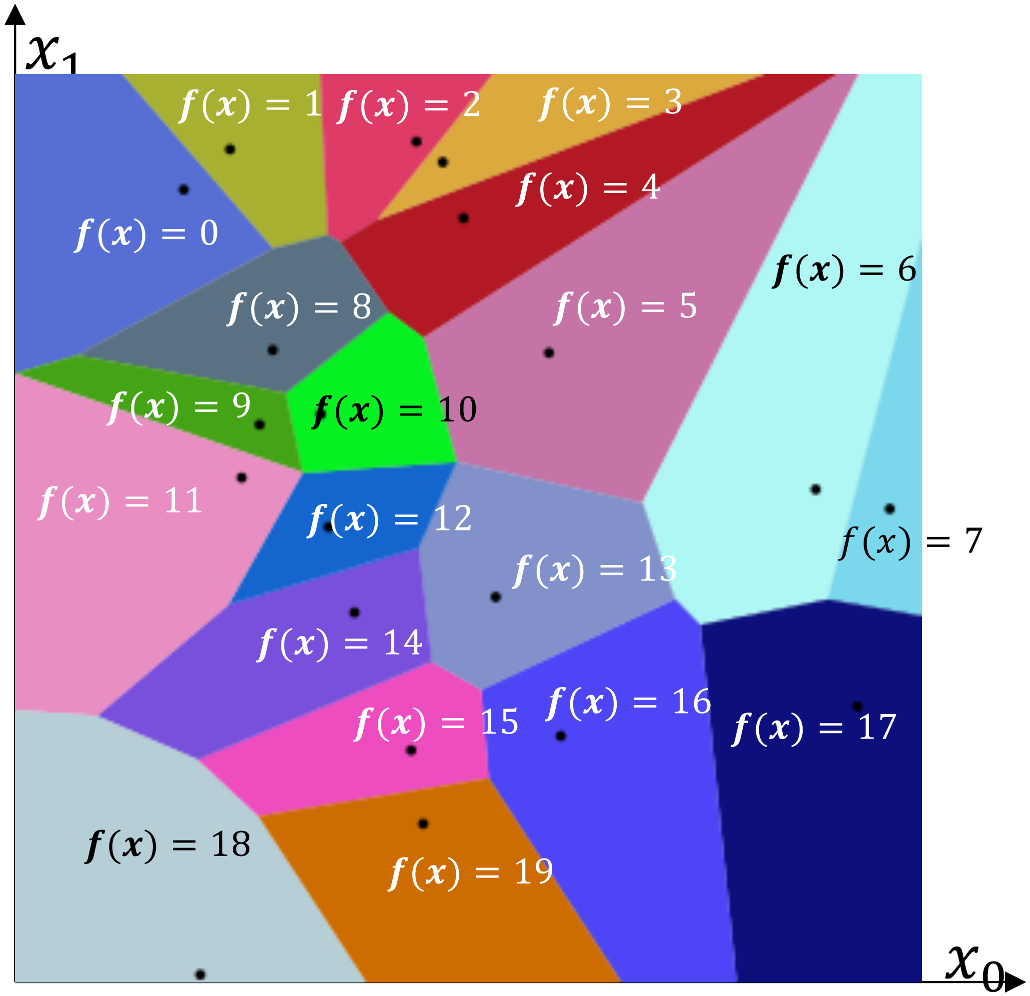
Multi-class linear classifier

In a general multi-class linear classifier,

$$f(x) = \underset{k}{\operatorname{argmax}}\, w_k @ x + b_k$$

The boundary between class k and class l is the line (or plane, or hyperplane) given by the equation

$$(w_l - w_k) @ x + (b_l - b_k) = 0$$

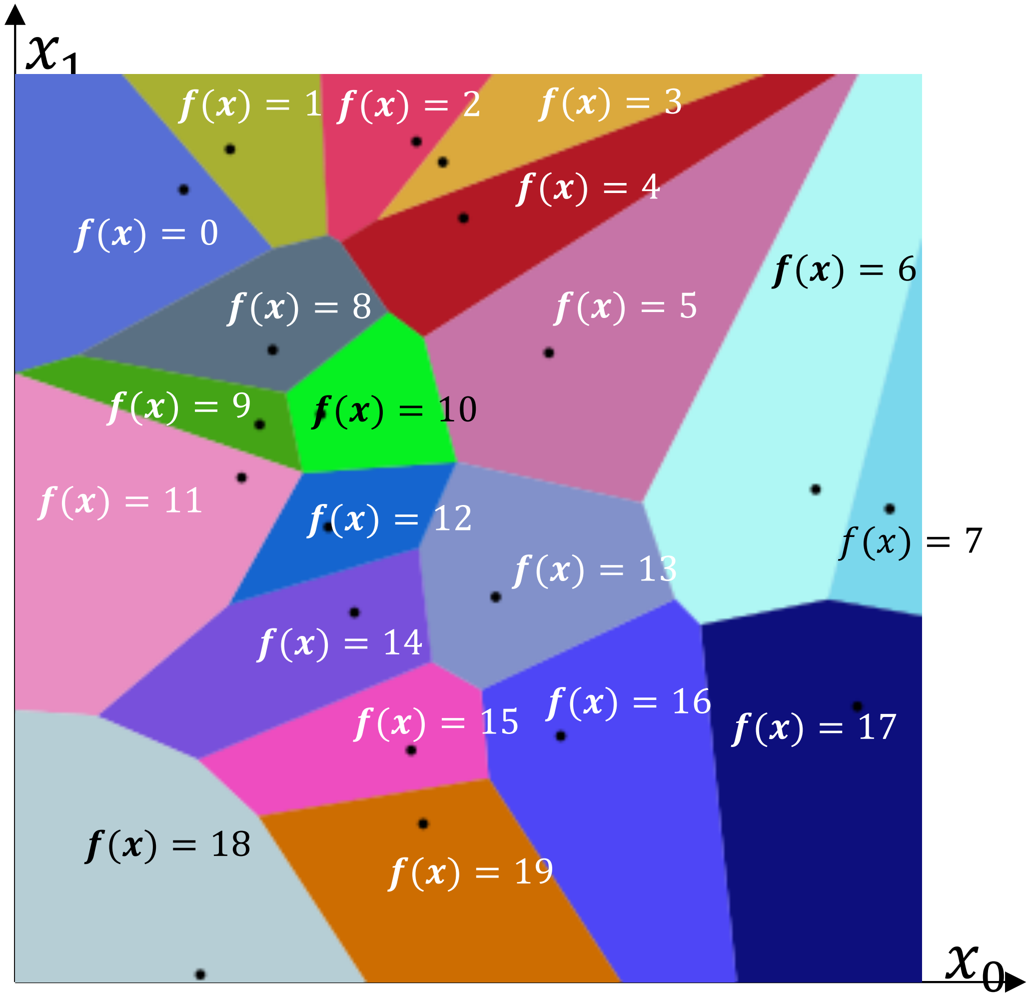


Voronoi regions

The classification regions in a linear classifier are called Voronoi regions.

A **Voronoi region** is a region that is

- Convex (if u and v are points in the region, then every point on the line segment $\overline{uv}$ connecting them is also in the region)
- Bounded by piece-wise linear boundaries



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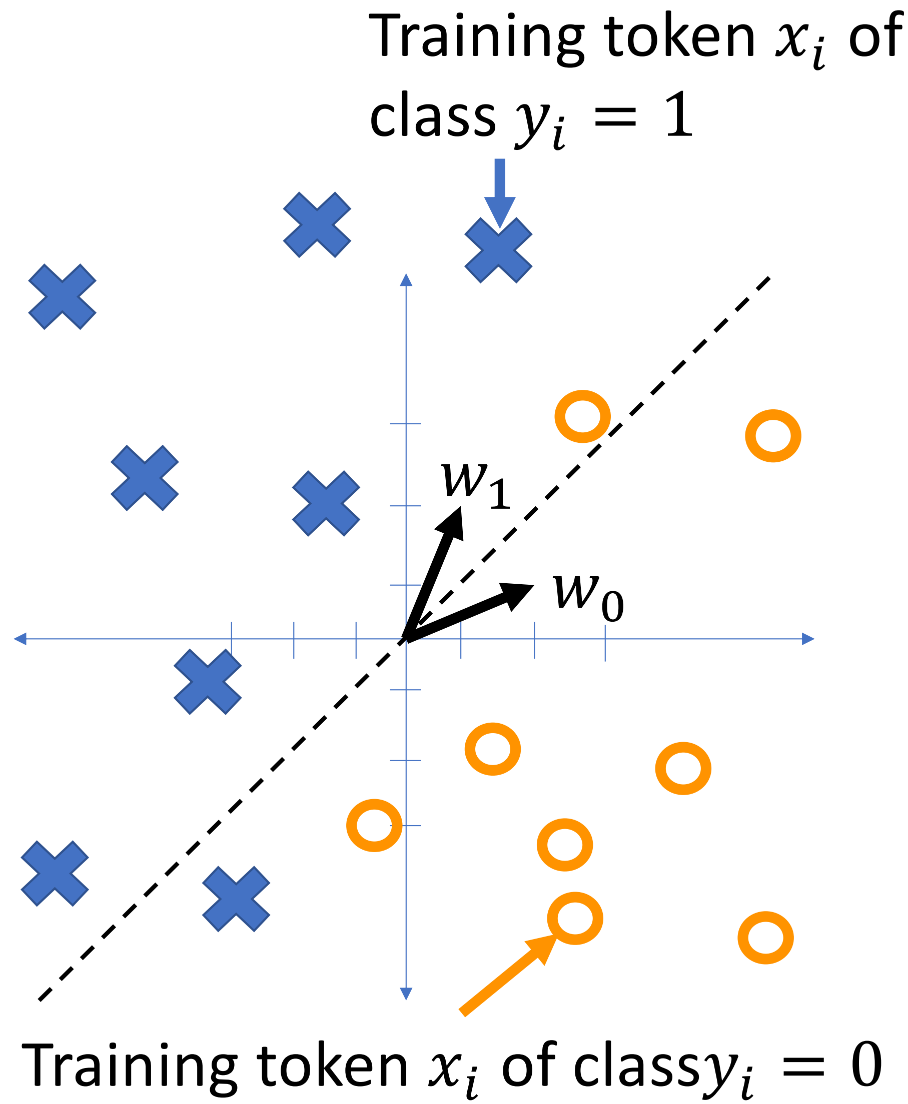
Gradient descent

Suppose we have training tokens (x_i, y_i), and we have some initial class vectors w_0 and w_1. We want to update them as

$$w_0 \leftarrow w_0 - \eta \nabla_{w_0} \mathcal{L}$$
$$w_1 \leftarrow w_1 - \eta \nabla_{w_1} \mathcal{L}$$

…where $\mathcal{L}$ is some loss function. What loss function makes sense?



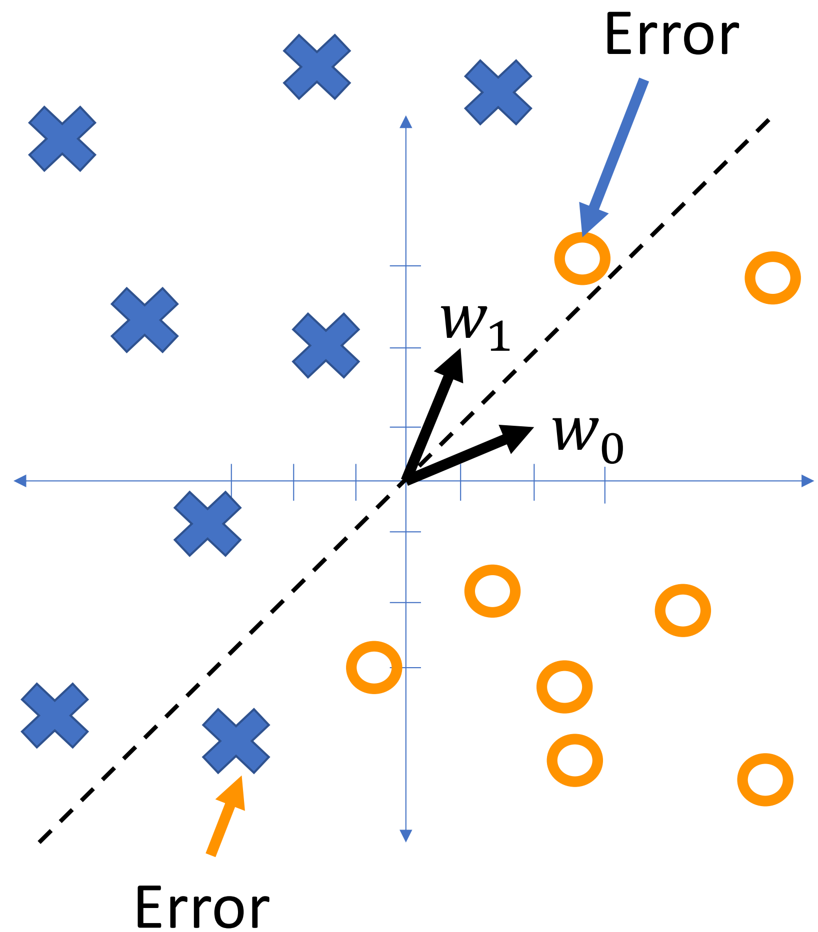
Zero-one loss function

The most obvious loss function for a classifier is its classification error rate,

$$\mathcal{L} = \frac{1}{n}\sum_{i=1}^{n} \ell(f(x_i), y_i)$$

Where $\ell(\hat{y}, y)$ is the zero-one loss function,

$$\ell(\hat{y}, y) = \begin{cases} 0 & \hat{y} = y \\ 1 & \hat{y} \neq y \end{cases}$$



Non-differentiable!

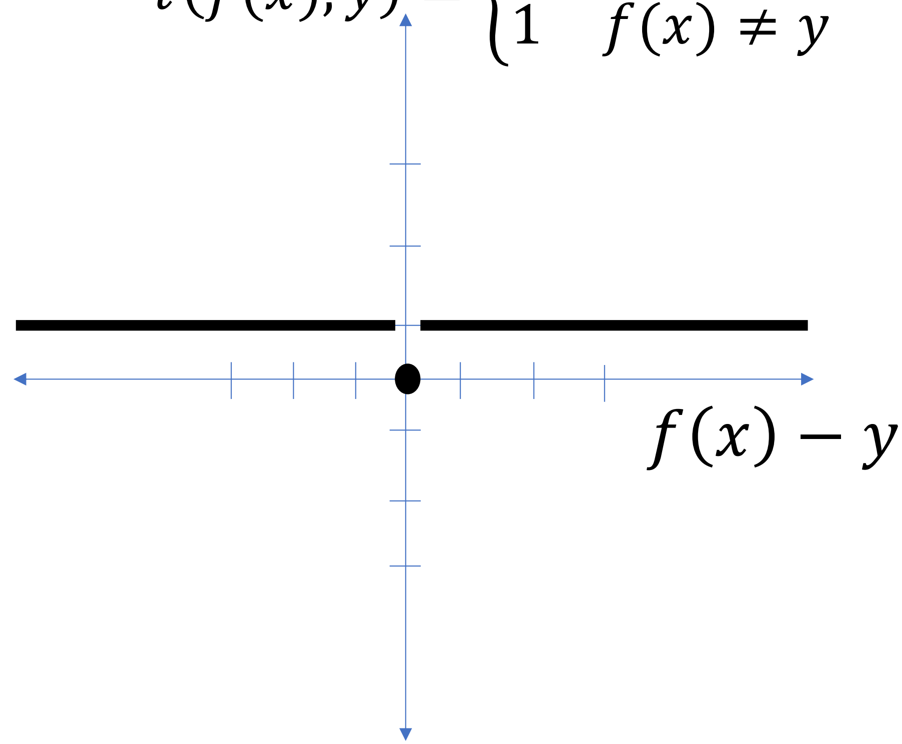
The problem with the zero-one loss function is that it's not differentiable:

$$\nabla_{w_0}\ell(f(x),y)$$

$$= \frac{\partial \ell(f(x),y)}{\partial f(x)} \nabla_{w_0} f(x)$$

$$= \begin{cases} 0 & f(x) \neq y \\ +\infty & f(x) = y^+ \\ -\infty & f(x) = y^- \end{cases}$$

$$\ell(f(x),y) = \begin{cases} 0 & f(x) = y \\ 1 & f(x) \neq y \end{cases}$$



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- Gradient descent
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One-hot vectors

A **one-hot vector** is a binary vector in which all elements are 0 except for a single element that's equal to 1.

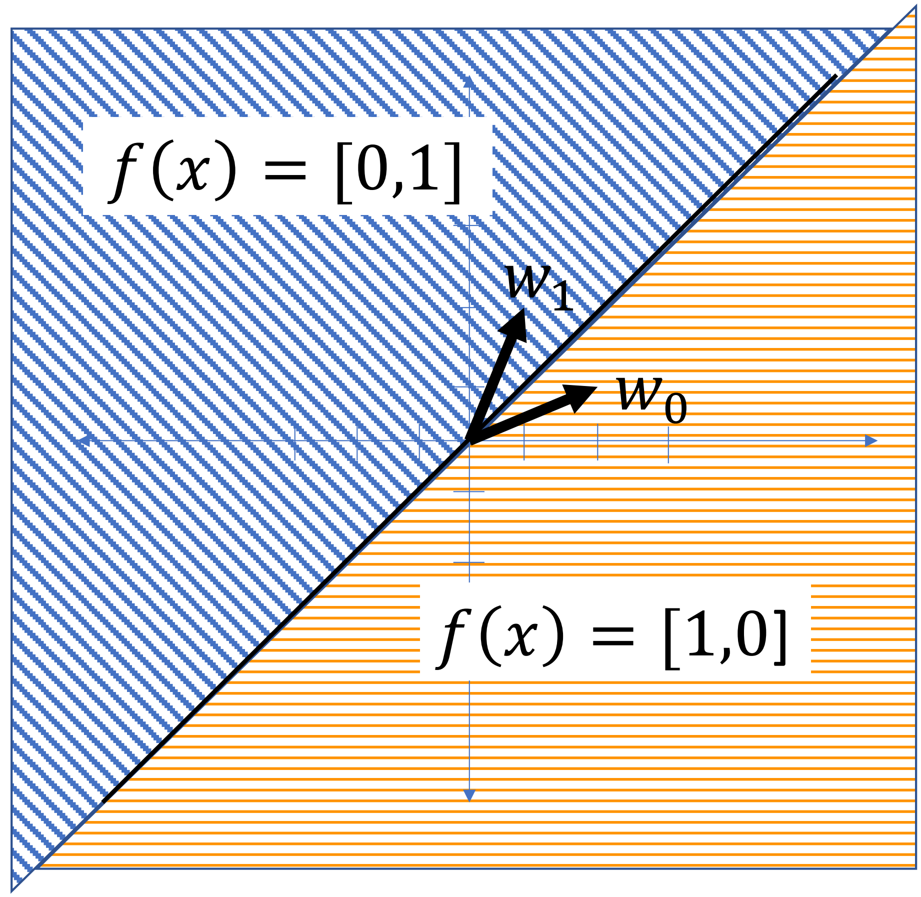
Example: Binary classifier

Consider the classifier

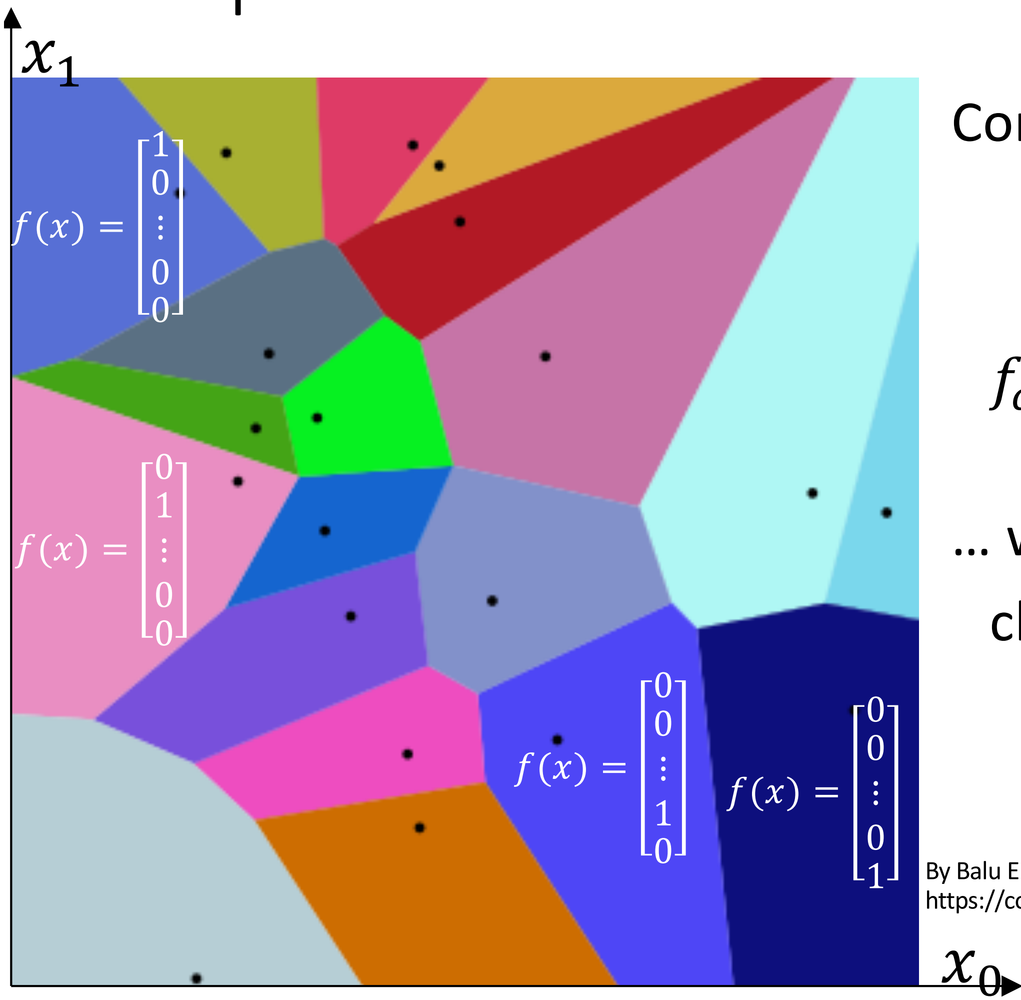
$$f(x_i) = [f_0(x_i), f_1(x_i)],$$

$$f_c(x_i) = \begin{cases} 1 & c = \underset{k}{\operatorname{argmax}}\, w_k@x + b_k \\ 0 & \text{otherwise} \end{cases}$$

… with two classes. Then the classification regions might look like this.



Example: Multi-Class



Consider the classifier

$$f(x_i) = [f_0(x_i), \ldots, f_{V-1}(x_i)]$$

$$f_c(x_i) = \begin{cases} 1 & c = \operatorname{argmax}_k w_k @ x + b_k \\ 0 & \text{otherwise} \end{cases}$$

… with 20 classes. Then some of the classifications might look like this.

Using one-hot vectors to calculate the loss

- Suppose that the output is a one-hot vector. Then the goal of the classifier is to set $f_c(x_i) = 1$ for the correct class, and $f_c(x_i) \approx 0$ for all others.
- We can measure this by a formula like:

$$\mathcal{L} = -\frac{1}{n}\sum_{i=1}^{n} \log f_{y_i}(x_i)$$

In words:

- choose the y_ith output of the classifier.
- If that output is $f_{y_i}(x_i) = 1$, then the loss is zero.
- If that output is $f_{y_i}(x_i) < 1$, then the loss is large (∞ if $f_{y_i}(x_i) = 0$).

Cross-entropy

This loss function,

$$\mathcal{L} = -\frac{1}{n}\sum_{i=1}^{n} \log f_{y_i}(x_i)\,,$$

is called cross-entropy. By measuring the negative log-probability of the correct class, we are measuring the **extra uncertainty** that is added to the system by **classification errors.**

Cross-entropy of a one-hot vector is still not differentiable!

Consider the classifier

$$f(x_i) = [f_0(x_i), \dots, f_{V-1}(x_i)]$$

$$f_c(x_i) = \begin{cases} 1 & c = \underset{k}{\operatorname{argmax}}\, w_k @ x + b_k \\ 0 & \text{otherwise} \end{cases}$$

Unfortunately, the cross-entropy of a one-hot vector is still not differentiable!

$$\mathcal{L} = -\log f_{y_i}(x_i) = \begin{cases} 0 & f_{y_i}(x_i) = 1 \\ \infty & f_{y_i}(x_i) = 0 \end{cases}$$

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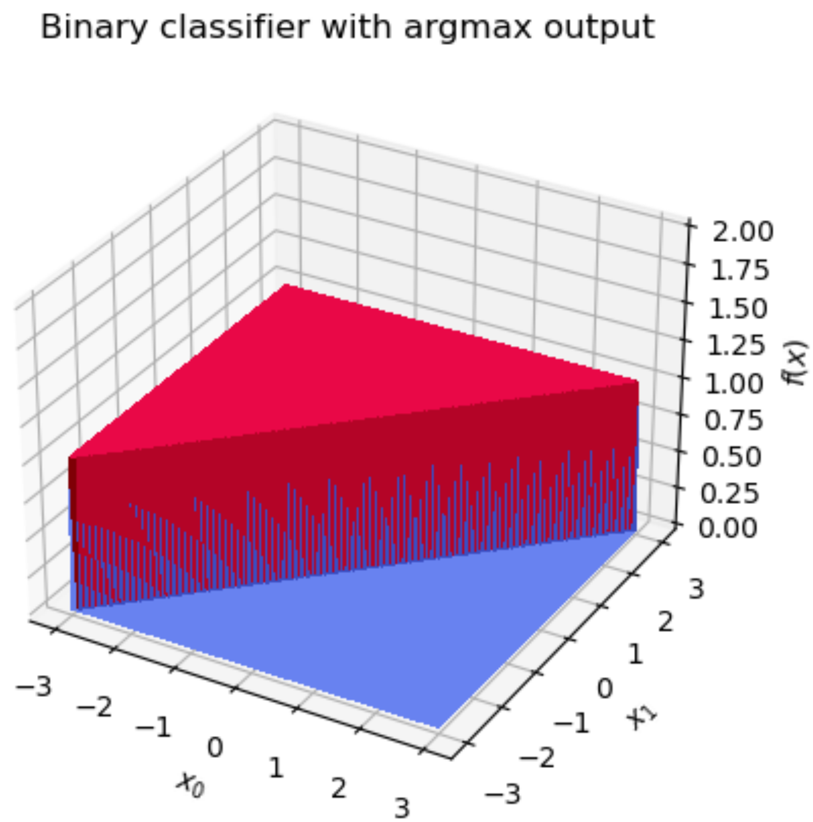
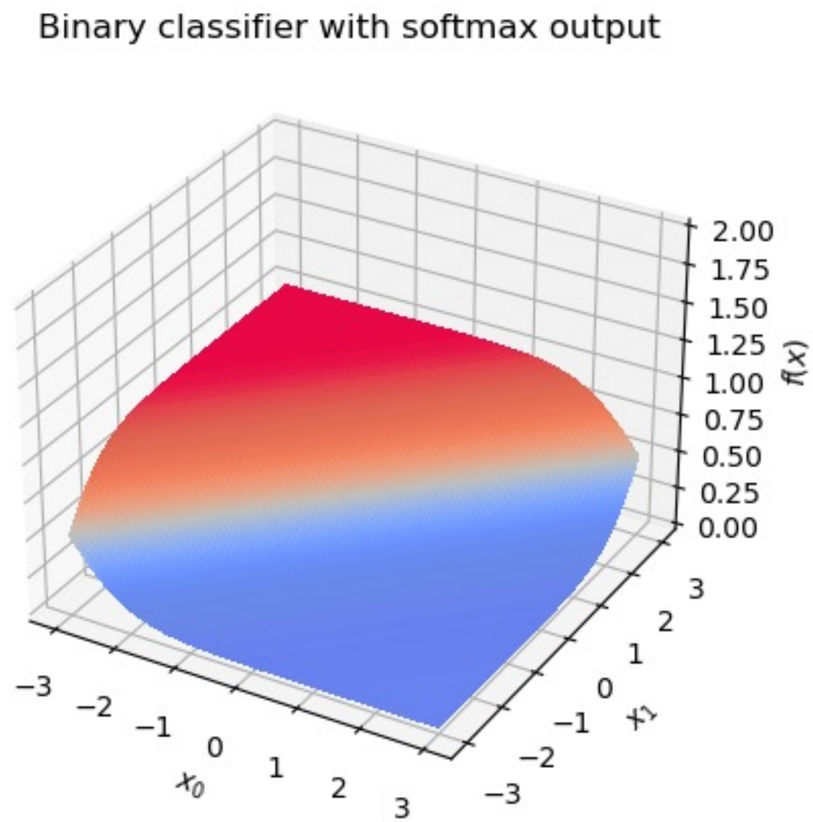
The problem with cross-entropy: $-\log 0 = \infty$

- Cross-entropy is a great loss function because $-\log 1 = 0$, so it measures no loss if the classifier has the right answer
- The problem is that $-\log 0 = \infty$, so if the classifier has the wrong answer, the loss function is unmeasurably huge



The solution: avoid 0-valued outputs

- The solution is to modify $f(x)$ so that it never outputs exactly 0
- Instead, we want $f(x)$ to approach 0 as the classifier gets more confident, but it should never actually reach zero



Argmax versus Softmax

The argmax version of the classifier is

$$f(x_i) = [f_0(x_i), \ldots, f_{V-1}(x_i)], \qquad f_c(x_i) = \begin{cases} 1 & c = \operatorname*{argmax}_k w_k @ x + b_k \\ 0 & \text{otherwise} \end{cases}$$

We can smooth it by using the softmax function, defined as

$$f(x_i) = [f_0(x_i), \ldots, f_{V-1}(x_i)], \qquad f_c(x_i) = \frac{\exp(w_c @ x + b_c)}{\sum_{k=0}^{V-1} \exp(w_k @ x + b_k)}$$

The softmax function

This is called the softmax function:

$$\text{softmax}(x_i) = [f_0(x_i), \dots, f_{V-1}(x_i)]$$

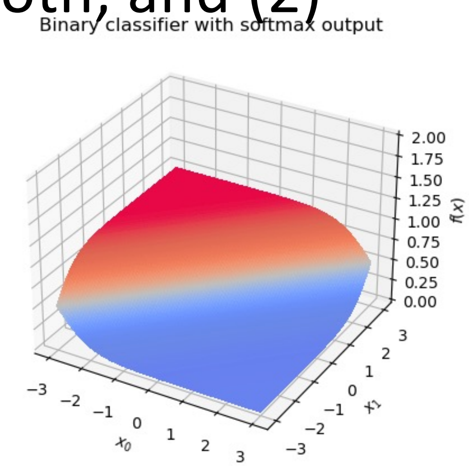
$$\underset{c}{\text{softmax}}(w@x + b) = \frac{\exp(w_c@x + b_c)}{\sum_{k=0}^{V-1} \exp(w_k@x + b_k)}$$

Key features of the softmax

$$\underset{c}{\operatorname{softmax}}(w@x + b) = \frac{\exp(w_c@x + b_c)}{\sum_{k=0}^{V-1} \exp(w_k@x + b_k)}$$

Notice that the softmax function is (1) smooth, and (2) behaves like a probability distribution:

- $0 < \underset{c}{\operatorname{softmax}}(w@x + b) < 1$
- $\sum_{c=0}^{V-1} \underset{c}{\operatorname{softmax}}(w@x + b) = 1$



Quiz

- Go to https://us.prairielearn.com/pl/course_instance/129874/assessment/2330383 and try the quiz

Gradient of the cross-entropy of the softmax

Consider the classifier

$$f_c(x_i) = \frac{\exp(w_c @ x + b_c)}{\sum_{k=0}^{V-1} \exp(w_k @ x + b_k)}$$

The softmax is smooth, so its logarithm is differentiable:

$$\mathcal{L} = -\log f_{y_i}(x_i) = -\left(w_{y_i} @ x + b_{y_i}\right) + \log \sum_{k=0}^{V-1} \exp(w_k @ x + b_k)$$

$$\nabla_{w_c} \mathcal{L} = \begin{cases} (f_c(x_i) - 1) x_i & c = y_i \\ f_c(x_i) x_i & \text{otherwise} \end{cases}$$

…is the same as the gradient of MSE for linear regression!

For linear regression, we had

$$\nabla_w \epsilon_i^2 = 2\epsilon_i x_i$$

For the softmax classifier with cross-entropy loss, we have

$$\nabla_{w_c} \mathcal{L} = \epsilon_{i,c} x_i$$

…where $\epsilon_{i,c}$ is the error of the cth output of the classifier:

$$\epsilon_{i,c} = \begin{cases} f_c(x_i) - 1 & c = y_i \text{ (output should be 1)} \\ f_c(x_i) - 0 & \text{otherwise(output should be 0)} \end{cases}$$

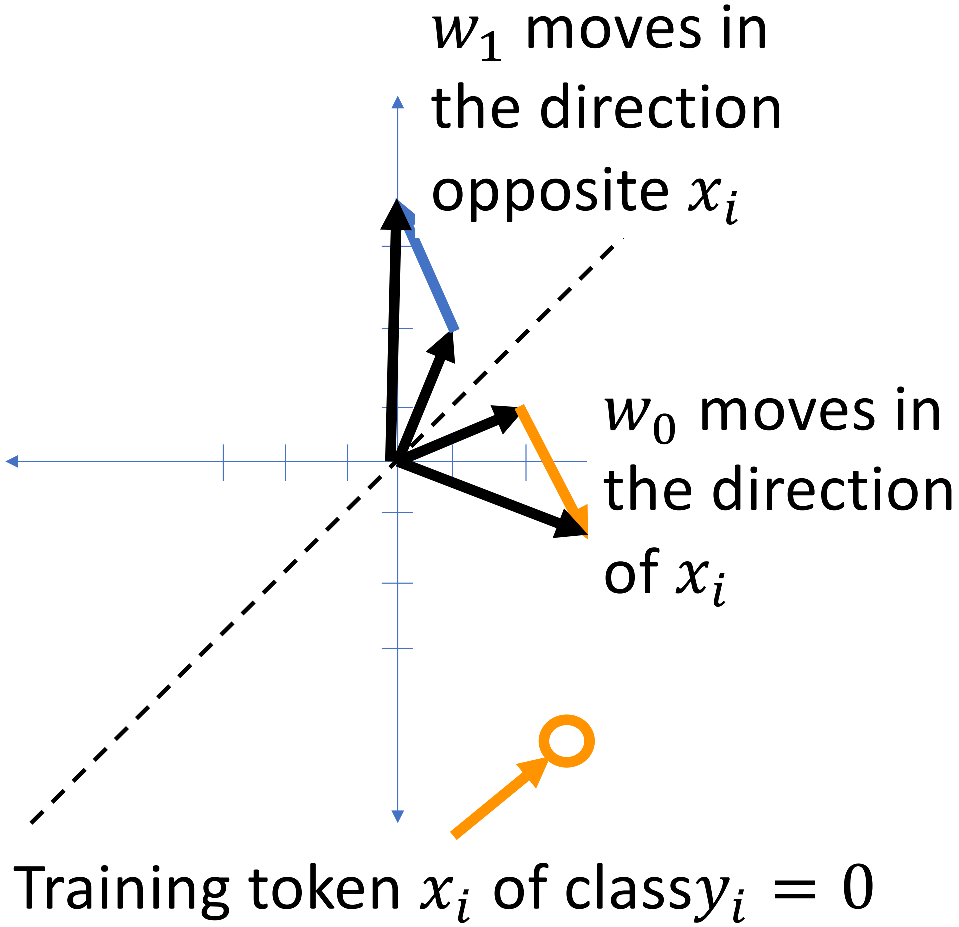
Stochastic gradient descent

Suppose we have a training token (x_i, y_i), and we have some initial class vectors w_c. Using softmax and cross-entropy loss, we can update the weight vectors as

$$w_c \leftarrow w_c - \eta \epsilon_{i,c} x_i$$

…where

$$\epsilon_{i,c} = \begin{cases} f_c(x_i) - 1 & c = y_i \\ f_c(x_i) - 0 & \text{otherwise} \end{cases}$$



Outline

- Linear Classifiers: $f(x) = \underset{k}{\operatorname{argmax}}\, w_k@x + b_k$
- Gradient descent: $w_c \leftarrow w_c - \eta \nabla_{w_c} \mathcal{L}$
- Cross-entropy: $\mathcal{L} = -\frac{1}{n}\sum_{i=1}^{n} \log f_{y_i}(x_i)$
- Softmax: $\underset{c}{\operatorname{softmax}}(w@x + b) \;= \frac{\exp(w_c@x+b_c)}{\sum_{k=0}^{V-1}\exp(w_k@x+b_k)}$
- Gradient of the cross-entropy of the softmax:

$$w_c \leftarrow w_c - \eta \epsilon_{i,c} x_i, \qquad \epsilon_{i,c} = \begin{cases} f_c(x_i) - 1 & c = y_i \text{ (output should be 1)} \\ f_c(x_i) - 0 & \text{otherwise(output should be 0)} \end{cases}$$