CS 440/ECE 448 Lecture 2: Random Variables

Mark Hasegawa-Johnson, 1/2023

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https://commons.wikimedia.org/wiki/File:6sided_dice.jpg

Outline

- Notation: Probability, Probability Mass, Probability Density
- Jointly random variables: joint, marginal, and conditional distributions
- Independence and Conditional independence
- Expectation
- Mean, Variance and Covariance
- Jointly Gaussian random variables

Notation: Probability

If an experiment is run an infinite number of times, the probability of event A is the fraction of those times on which event A occurs.

Axiom 1: every event has a non-negative probability.

$$\Pr(A) \geq 0$$

Axiom 2: If an event always occurs, we say it has probability 1.

$$\Omega = \begin{cases} T & \text{always} \\ F & \text{never} \end{cases}$$

$$\Pr(\Omega = \mathrm{T}) = 1$$

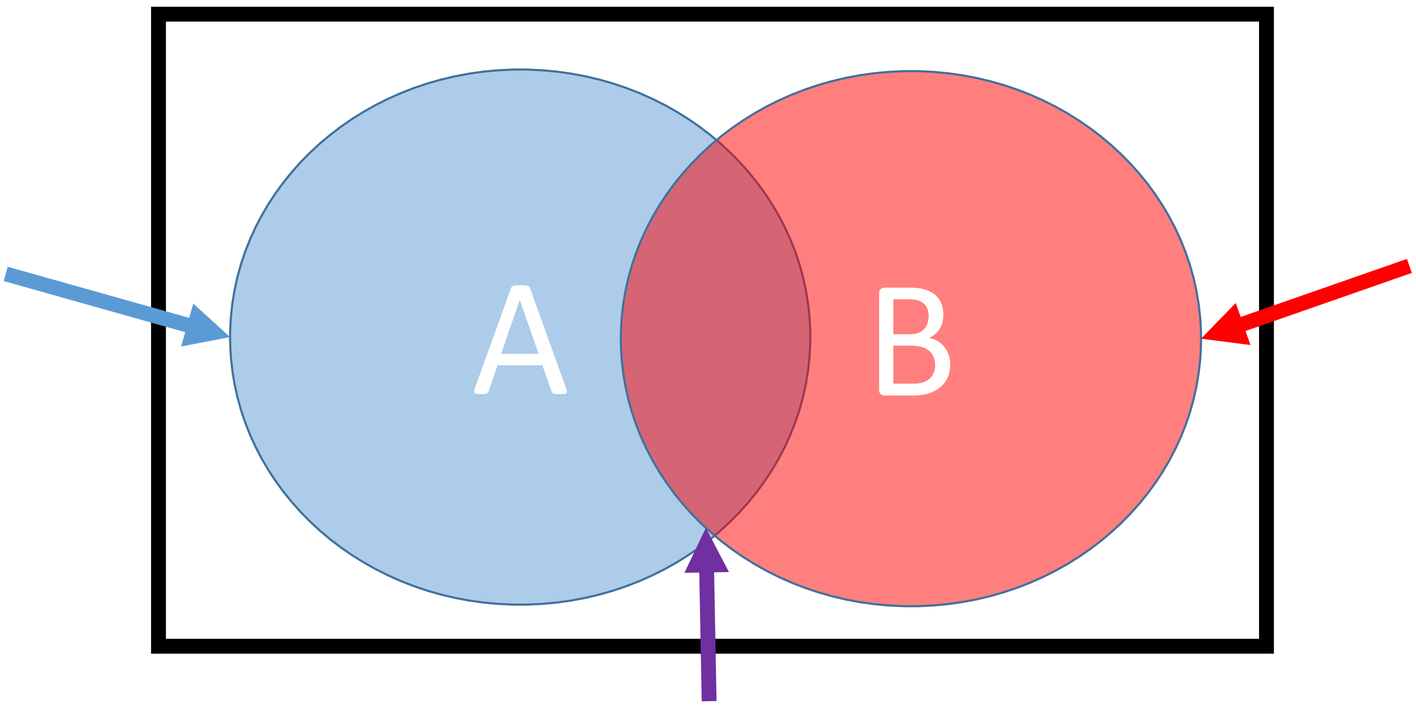
Axiom 3: probability measures behave like set measures.

$$\Pr(A \lor B) = \Pr(A) + \Pr(B) - \Pr(A \land B)$$

Axiom 3: probability measures behave like set measures.

Area of the whole rectangle is $P(\Omega = \mathrm{T}) = 1$.

Area of this circle is $P(A)$.



Area of this circle is $P(B)$.

Area of their intersection is $P(A \wedge B)$.
Area of their union is $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

Notation: Random Variables

A **random variable** is a function that summarizes the output of an experiment. We use **capital letters** to denote random variables.

- Example: every Friday, Maria brings a cake to her daughter's pre-school. X is the number of children who eat the cake.

We use a **small letter** to denote a particular **outcome** of the experiment.

- Example: for the last three weeks, each week, 5 children had cake, but this week, only 4 children had cake. Estimate $P(X = x)$ for all possible values of x.

Notation: $P(X = x)$ is a number, but $P(X)$ is a distribution

- $P(X = 4)$ or $P(4)$ is the probability mass or probability density of the outcome "$X = 4$." For example:

$$P(X = 4) = \frac{1}{4}$$

- $P(X)$ is the complete **distribution**, specifying $P(X = x)$ for all possible values of x. For example:

$P(X) =$

x	**4**	**5**
$P(x)$	$\frac{1}{4}$	$\frac{3}{4}$

Discrete versus Continuous RVs

- X is a **discrete random variable** if it can only take countably many different values.
  - Example: X is the number of people living in a randomly selected city

$$X \in \{1,2,3,4, \dots\}$$

  - Example: X is the first word on a randomly selected page

$$X \in \{\text{the}, \text{and}, \text{of}, \text{bobcat}, \dots\}$$

  - Example: X is the next emoji you will receive on your cellphone

$X \in \{$😀, 😃, 😄, 😁, 😆, 😅, 🤣, …$\}$

- X is a **continuous random variable** if it can take uncountably many different values
  - Example: X is the energy of the next object to collide with Earth

$$X \in \mathbb{R}^{+} \text{ (the set of all positive real numbers)}$$

Probability Mass Function (pmf) is a type of probability

- If X is a **discrete random variable,** then $P(X)$ is its **probability mass function (pmf)**.
- A probability mass is just a probability. $P(X = x)$ is the just the probability of the outcome "$X = x$." Thus:

$$0 \leq P(X = x)$$

$$1 = \sum_{x} P(X = x)$$

Probability Density Function (pdf) is NOT a probability

- If X is a **density random variable,** then $P(X)$ is its **probability density function (pdf)**.
- A probability density is NOT a probability. Instead, we define it as a density ($P(X = x) = \frac{d}{dx}\Pr(X \leq x)$). Thus:

$$0 \leq P(X = x)$$

$$1 = \int_{-\infty}^{\infty} P(X = x)dx$$

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Jointly Random Variables

- Two or three random variables are “jointly random” if they are both outcomes of the same experiment.
- For example, here are the temperature (Y, in °C), and precipitation (X, symbolic) for six days in Urbana:

	X=Temperature (°C)	**Y=Precipitation**
January 11	4	cloud
January 12	1	cloud
January 13	-2	snow
January 14	-3	cloud
January 15	-3	clear
January 16	4	rain

Joint Distributions

Based on the data on prev slide, here is an estimate of the joint distribution of these two random variables:

	snow	**rain**	**cloud**	**clear**
-3	0	0	1/6	1/6
-2	1/6	0	0	0
1	0	0	1/6	0
4	0	1/6	1/6	0

Marginal Distributions

Suppose we know the joint distribution $P(X, Y)$. We want to find the two **marginal distributions** $P(X)$:

- If the unwanted variable is discrete, we marginalize by adding:

$$P(X) = \sum_{y} P(X, Y = y)$$

- If the unwanted variable is continuous, we marginalize by integrating:

$$P(X) = \int P(X, Y = y)dy$$

Marginal Distributions

Here are the marginal distributions of the two weather variables:

	snow	rain	cloud	clear
-3	0	0	1/6	1/6
-2	1/6	0	0	0
1	0	0	1/6	0
4	0	1/6	1/6	0

P(X)
1/3
1/6
0
1/3

P(Y)	1/6	1/6	1/2	1/6

Joint and Conditional distributions

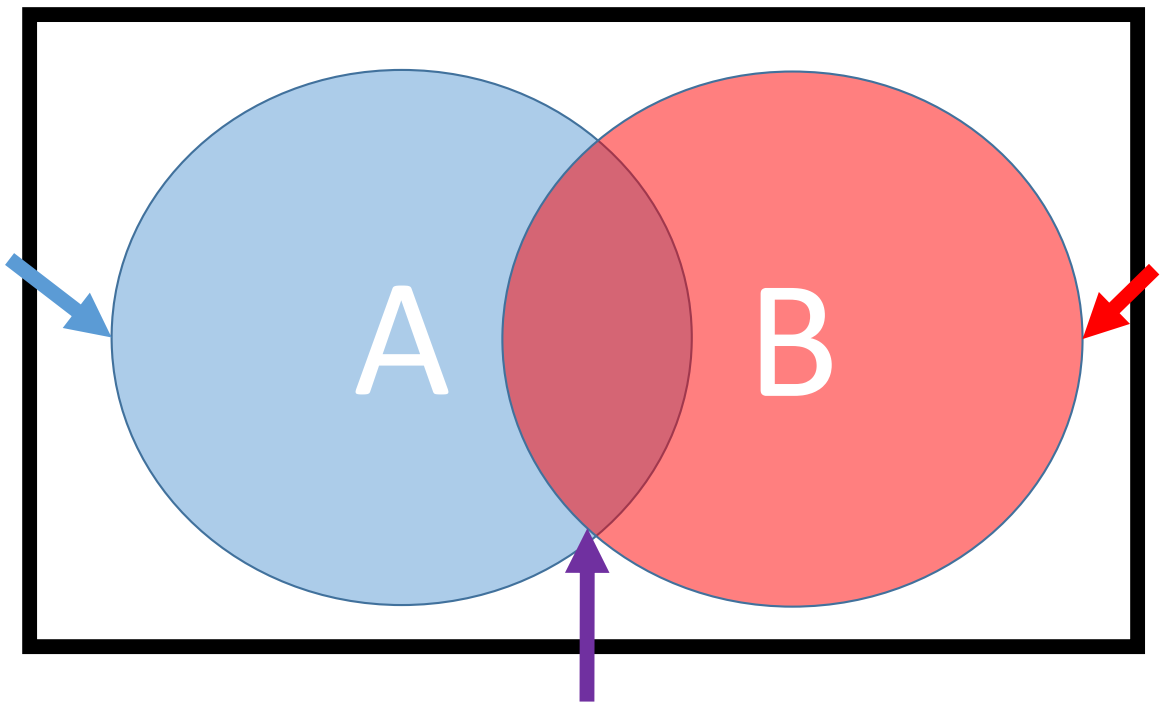
- $P(X,Y)$ is the probability (or pdf) that $X = x$ and $Y = y$, over all x and y. This is called their **<u>joint distribution</u>**.

- $P(Y|X)$ is the probability (or pdf) that $Y = y$ happens, given that $X = x$ happens, over all x and y. This is called the **<u>conditional distribution</u>** of Y given X.

Joint probabilities are usually given in the problem statement

Area of the whole rectangle is $\Pr(\text{True}) = 1$.

Suppose $\Pr(A) = 0.4$



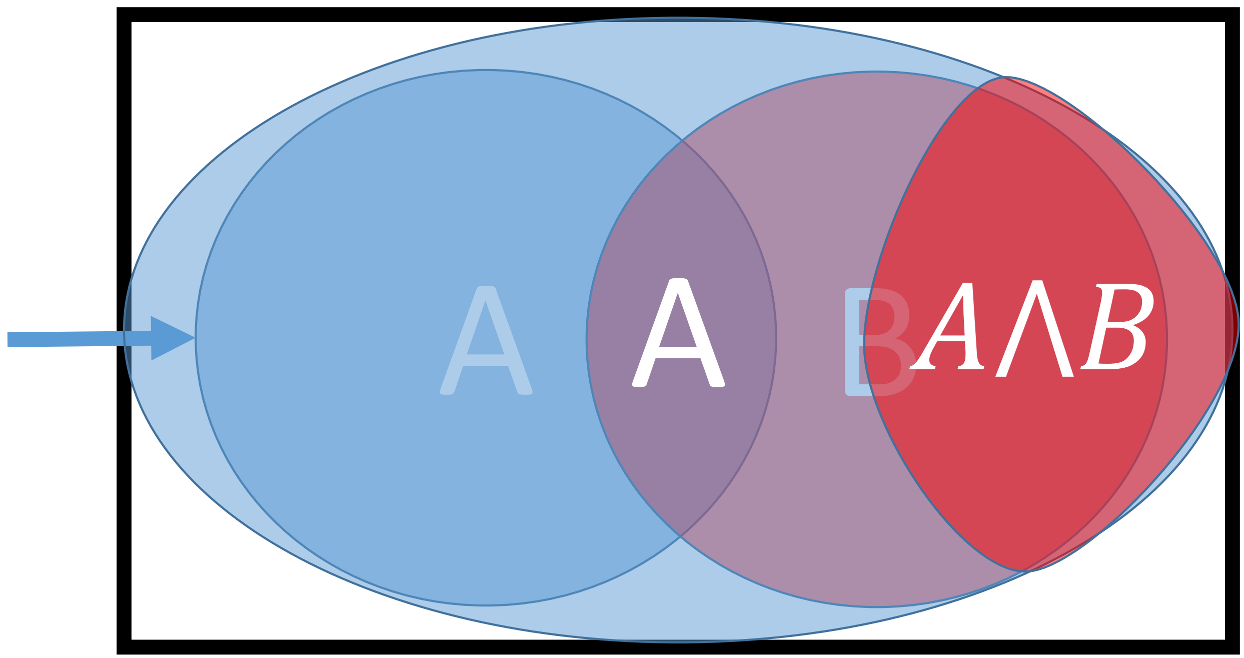
Suppose $\Pr(B) = 0.2$

Suppose $\Pr(A \wedge B) = 0.1$

Conditioning events change our knowledge!
For example, given that A is true…

Most of the events in this rectangle are no longer possible!

Only the events inside this circle are now possible.



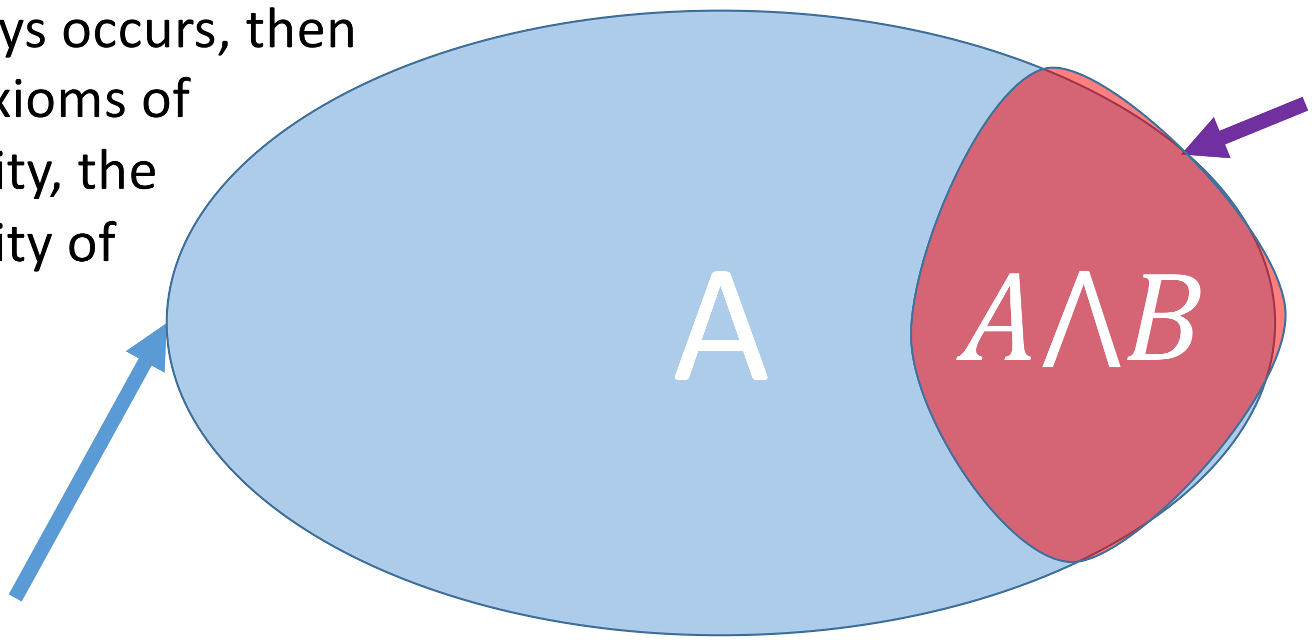
Conditioning events change our knowledge!
For example, given that A is true…

The probability of B, given A, is the size of the event $A \wedge B$, expressed as a fraction of the size of the event A:

$$\Pr(B|A) = \frac{\Pr(A \wedge B)}{\Pr(A)}$$

If A always occurs, then by the axioms of probability, the probability of
A=T
is 1.
We can say that

$\Pr(A|A) = 1.$



Joint and Conditional distributions of random variables

- $P(X, Y)$ is the **joint probability distribution** over all possible outcomes $P(X = x, Y = y)$.
- $P(X|Y)$ is the **conditional probability distribution** of outcomes $P(X = x|Y = y)$.
- The **conditional** is the **joint** divided by the **marginal:**

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Conditional is the joint divided by the marginal:

$$P(X|Y = \text{cloud}) = \frac{P(X, Y = \text{cloud})}{P(Y = \text{cloud})} = \frac{\begin{bmatrix}\frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6}\end{bmatrix}}{1/2}$$

	snow	**rain**	**cloud**	**clear**
-3	0	0	1/6	1/6
-2	1/6	0	0	0
1	0	0	1/6	0
4	0	1/6	1/6	0

P(X|Y=cloud)
1/3
0
1/3
1/3

Joint = Conditional×Marginal

$$P(X,Y) = P(X|Y)P(Y)$$

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Independent Random Variables

Two random variables are said to be independent if:

$$P(X|Y) = P(X)$$

In other words, knowing the value of Y tells you nothing about the value of X.

… and a more useful definition of independence…

Plugging the definition of independence,

$$P(X|Y) = P(X),$$

…into the "Joint = Conditional×Marginal" equation,

$$P(X,Y) = P(X|Y)P(Y)$$

…gives us a more useful definition of independence.

Definition of Independence: Two random variables, X and Y, are independent if and only if

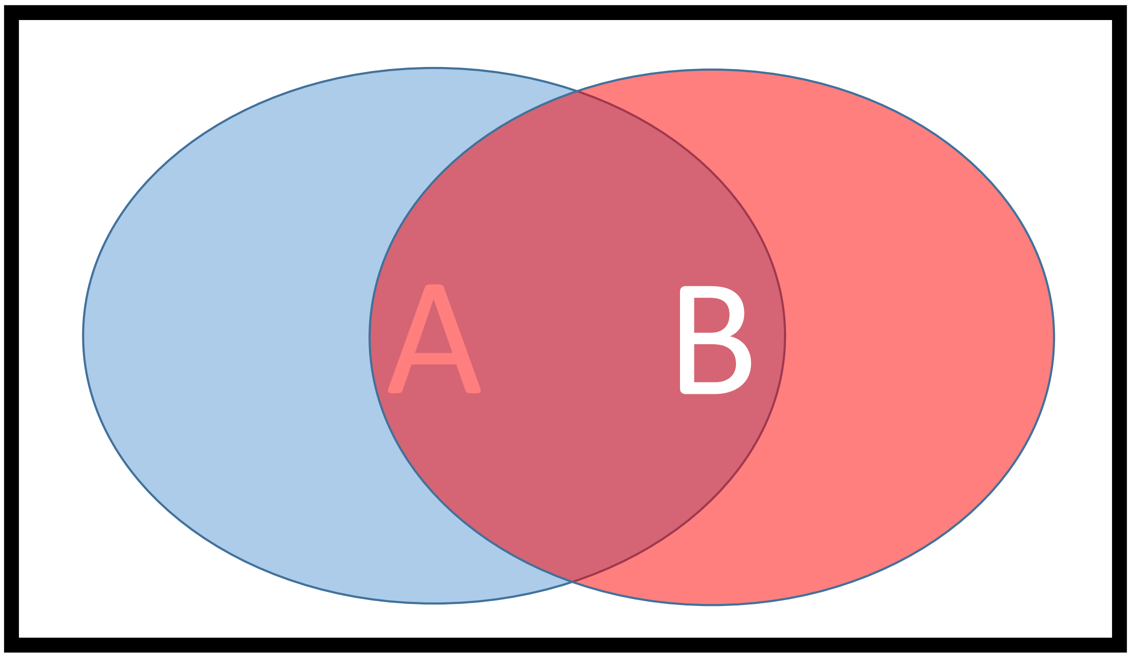
$$P(X,Y) = P(X)P(Y)$$

Independent events

Independent events occur with equal probability, regardless of whether or not the other event has occurred:

$$\Pr(A|B) = \Pr(A)$$

$$\Pr(A \wedge B) = \Pr(A)\Pr(B)$$



Conditionally Independent Random Variables

Two random variables X and Y are said to be conditionally independent given knowledge of Z if:

$$P(X|Y,Z) = P(X|Z)$$

In other words, if you know the value of Z, then also knowing the value of Y tells you nothing ***new*** about the value of X.

… and a more useful definition of conditional independence…

Plugging the definition of conditional independence,

$$P(X|Y,Z) = P(X|Z),$$

…into the "Joint = Conditional×Marginal" equation,

$$P(X,Y,Z) = P(X|Y,Z)P(Y|Z)P(Z)$$

…gives us a more useful definition of conditional independence.

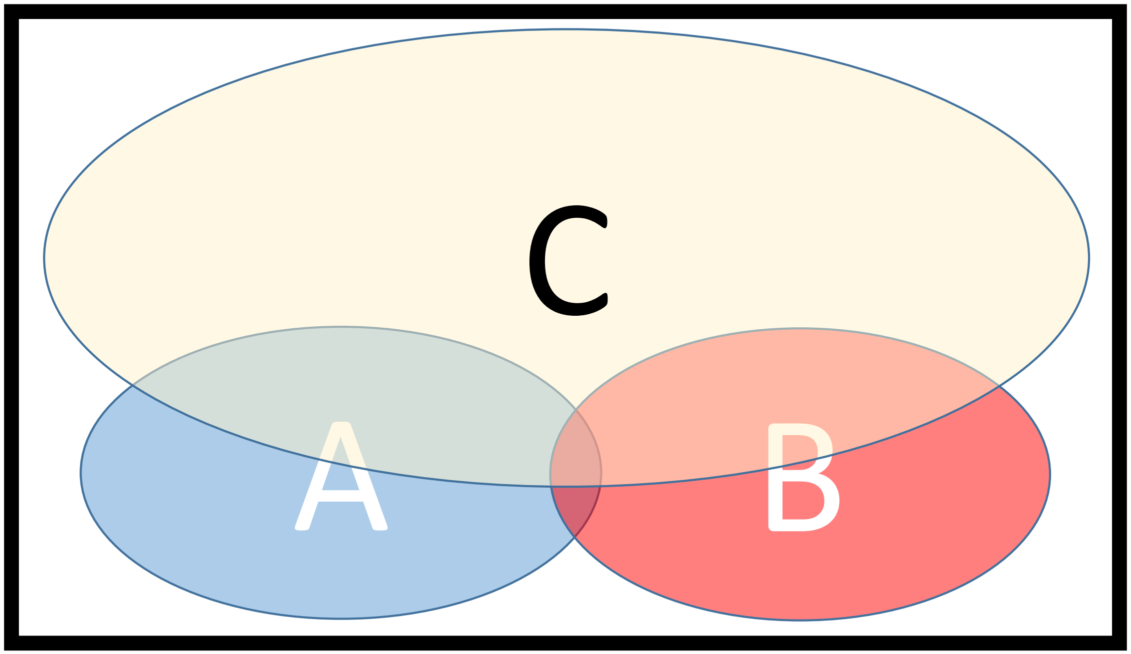
Definition of Conditional Independence: Two random variables, X and Y, are conditionally independent given Z if and only if

$$P(X,Y,Z) = P(X|Z)P(Y|Z)P(Z)$$

Conditionally independent events

Events A and B are conditionally independent, given C, if and only if

$$\Pr(A|B, C) = \Pr(A|C)$$
$$\Pr(A \wedge B|C) = \Pr(A|C)\Pr(B|C)$$



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Expectation

- The expected value of a function is its weighted average, weighted by its pmf or pdf.
- If X and Y are discrete, then

$$E[f(X,Y)] = \sum_{x,y} f(x,y)P(X = x, Y = y)$$

- If X is continuous, then

$$E[f(X,Y)] = \iint_{-\infty}^{\infty} f(x,y)P(X = x, Y = y)dxdy$$

Quiz question

Go to https://us.prairielearn.com/pl/course_instance/129874/

Take the quiz called “20-Jan”

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Mean and Variance

- The mean of a random variable is its expected value:

$$E[X] = \sum_x xP(X = x)$$

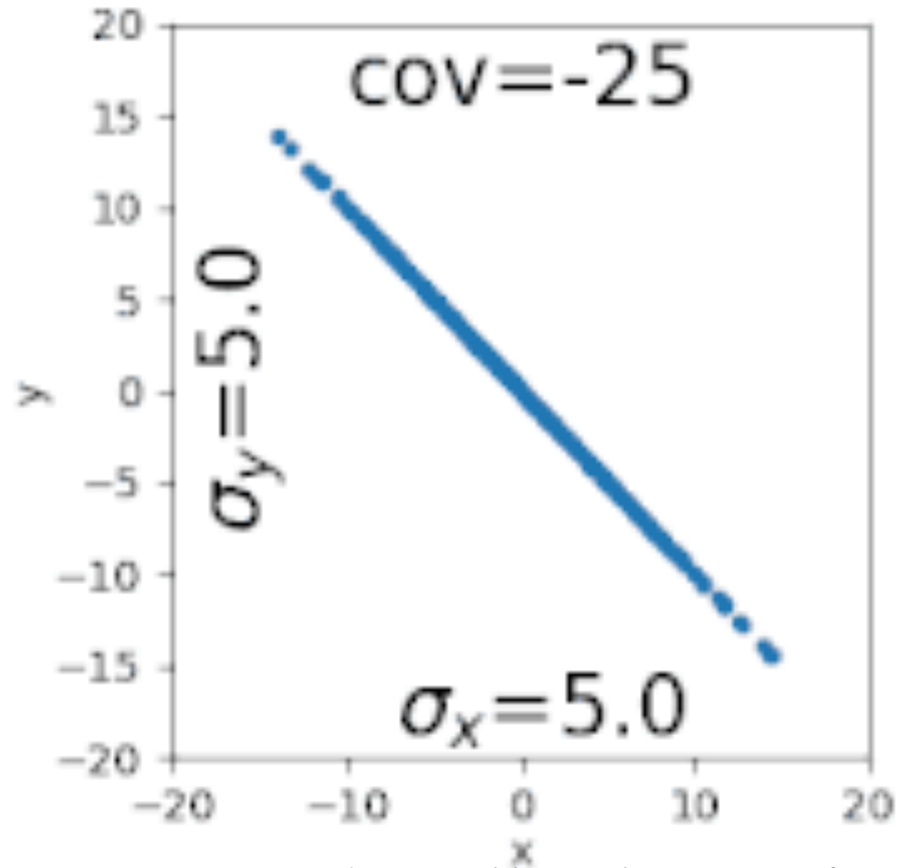
- The variance of a random variable is the expected value of its squared deviation from its mean:

$$\mathrm{Var}(X) = E[(X - E[X])^2] = \sum_x (x - E[X])^2 P(X = x)$$

Covariance

The covariance of two random variables is the expected product of their deviations:

$$\text{Covar}(X, Y)$$
$$= E[(X - E[X])(Y - E[Y])]$$



Two zero-mean random variables, with variances of 25, and with various values of covariance.
Public domain image,
https://commons.wikimedia.org/wiki/File:Varianz.gif

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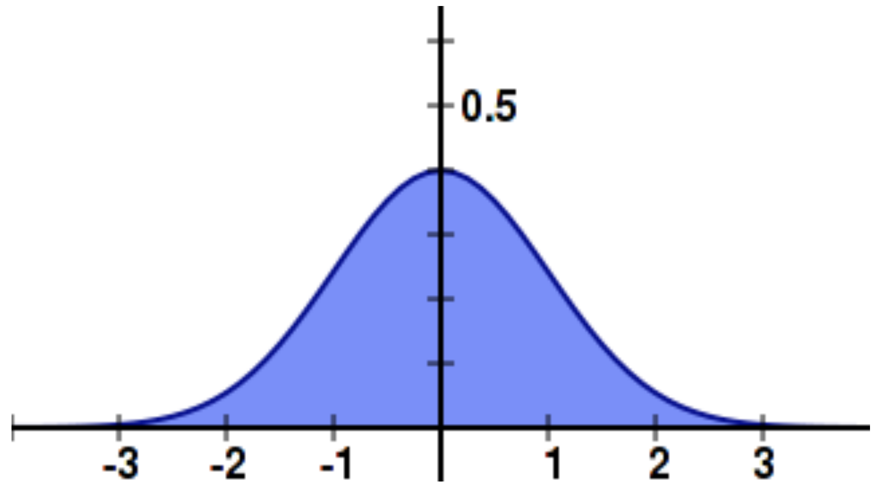
Gaussian Random Variable

If X is the average of many independent identically distributed random variables, it tends to have the following pdf, called a "Gaussian" or "normal" pdf:

$$P(X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

X

…where $\mu = E[X]$ and $\sigma^2 = \mathrm{Var}(X)$.

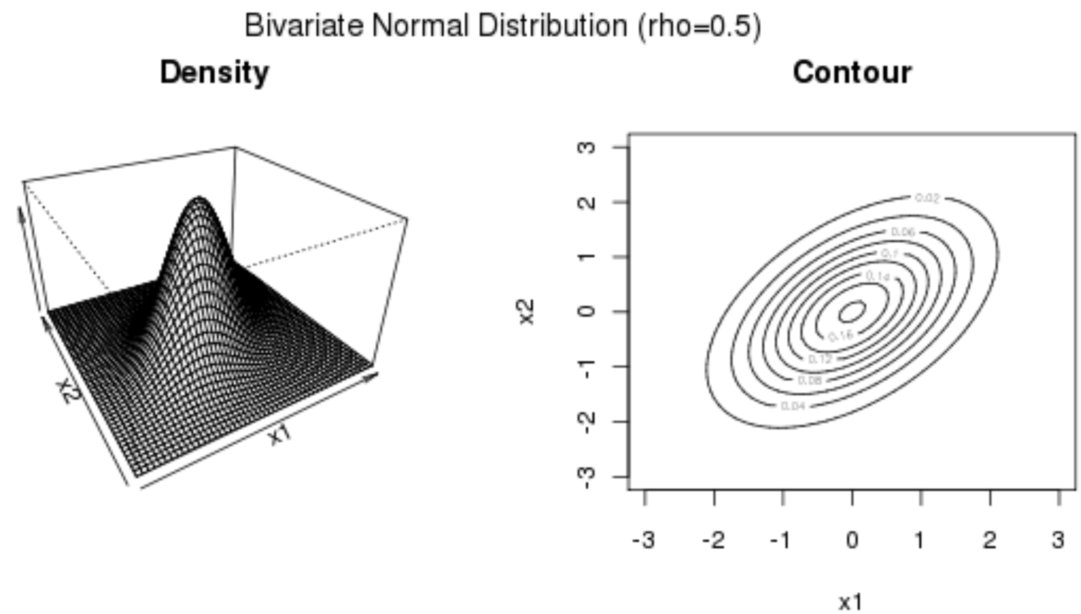


"The famous bell curve."
CC-SA 3.0, Philip Rideout, 2012

Jointly Gaussian Random Variables

X and Y are jointly Gaussian if and only if

- $P(X)$ is a Gaussian pdf, and
- $P(Y|X)$ is a Gaussian pdf.



Summary

- Probability, Probability Mass, Probability Density
  - X is a random variable, x is its instance value
- Jointly random variables: joint, marginal, and conditional distributions
  - Joint = Conditional × Marginal
- Independence and Conditional independence
  - X and Y conditionally independent given Z: $P(X,Y,Z) = P(X|Z)P(Y|Z)P(Z)$
- Expectation

$$E[f(X,Y)] = \sum_{x,y} f(x,y)P(X = x, Y = y)$$

- Mean, Variance and Covariance

$$\mathrm{Covar}(X,Y) = E[(X - E[X])(Y - E[Y])]$$

- Jointly Gaussian random variables
  - $P(X)$ is a Gaussian pdf, and $P(Y|X)$ is a Gaussian pdf.