Lecture 17: The “animal kingdom” of heuristics: Admissible, Consistent, Zero, Relaxed, Dominant

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Title image: Peaceable Kingdom by Edward Hicks, National Gallery of Art, Washington, DC
Outline of lecture

1. Admissible heuristics
2. Consistent heuristics
3. The zero heuristic: Uniform Cost Search
4. Relaxed heuristics
5. Dominant heuristics
**A* Search**

**Definition: A* SEARCH**

- If $h(n)$ is **admissible** ($d(n) \geq h(n)$), and
- if the frontier is a priority queue sorted according to $g(n) + h(n)$, then
- the FIRST path to goal uncovered by the tree search, path $m$, is guaranteed to be the **SHORTEST** path to goal

$$h(n) + g(n) \geq c(m) \text{ for every node } n \text{ that is not on path } m$$
Bad interaction between A* and the explored set

Explored Set:
empty

Frontier
S: \(g(S)+h(S)=2\), \(g(S)=0\), parent=none

Expand: S, put its children A and B on the frontier.
Bad interaction between A* and the explored set

Explored Set:
S

Frontier
A: g(A)+h(A)=5, g(A)=1, parent=S
B: g(B)+h(B)=2, g(B)=1, parent=S

Expand: B, put its child C on the frontier.
Bad interaction between A* and the explored set

Explored Set:
S, B

Frontier
A: g(A)+h(A)=5, g(A)=1, parent=S
C: g(C)+h(C)=4, g(C)=3, parent=B

Expand: C, put its child G on the frontier.
Bad interaction between A* and the explored set

Explored Set:
S, B, C

Frontier
A: $g(A)+h(A)=5$, $g(A)=1$, parent=S
G: $g(G)+h(G)=6$, $g(G)=6$, parent=C

Expand: A. But we can’t put its child, C, on the frontier, because C is already in the explored set!
Bad interaction between A* and the explored set

Explored Set:
S, B, C

Frontier
G: $g(G) + h(G) = 6$, $g(G) = 6$, parent = C

Expand: G. Return the path SBCG, with cost 6. OOPS!
Why did this happen?

• Well, because we used an explored set instead of an explored dict.
  • explored dict lists the $h(n)+g(n)$ for each explored state
  • If the same state shows up later, with lower $h(n)+g(n)$, then put it back on the frontier.
  • An explored set undermines A*, but an explored dict works just fine.
• But actually, why did the higher-cost path SBC get explored before the lower-cost path SAC?
  • That never happens for goal. An admissible heuristic guarantees that the first time you pop Goal from the frontier, it will have its lowest cost.
  • Can we make the same idea true for every state, not just the goal state?
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Consistent (monotonic) heuristic

**Definition:** A **consistent heuristic** is one for which, for every pair of nodes $n$ and $p$ such that $d(n) \geq d(p)$, $d(n) - d(p) \geq h(n) - h(p)$. 

In words: the **distance** between any pair of nodes is **greater than or equal to** the **difference** in their heuristics.
A* with an inconsistent heuristic

Inconsistent because:

\[ d(A) - d(C) = 1 \]

but

\[ h(A) - h(C) = 3 \]

To fix this, we need to reduce \( h(A) \) so that \( h(A) - h(C) \leq 1 \).
A* with a **consistent** heuristic

Consistent because:

\[ d(A) - d(C) = 1 \]

and

\[ h(A) - h(C) = 0 \]

Similarly

\[ h(n) - h(p) \leq d(n) - d(p) \]

...for every pair of nodes \( n \) and \( p \) such that \( d(n) \geq d(p) \).
A* with an inconsistent heuristic

Explored Set
S, B

Frontier
A: $g(A) + h(A) = 5$, $g(A) = 1$, parent=S
C: $g(C) + h(C) = 4$, $g(C) = 3$, parent=B

Expand: C
A* with a **consistent** heuristic

Explored Set
S, B

Frontier
A: \( g(A) + h(A) = 2 \), \( g(A) = 1 \), parent=S
C: \( g(C) + h(C) = 4 \), \( g(C) = 3 \), parent=B

Expand: A
A* with a **consistent** heuristic

Explored Set
S, B, **A**

Frontier
C: $g(C)+h(C)=4$, $g(C)=3$, parent=B
C: $g(C)+h(C)=\text{3}$, $g(C)=2$, parent=**A**

Expand: the copy of **C** that has **A** as its parent.
A* with a **consistent** heuristic

Explored Set
S, B, A, C

Frontier
C: $g(C)+h(C)=4$, $g(C)=3$, parent=B
G: $g(G)+h(G)=5$, parent=C

Expand: The copy of C that has B as its parent. But C is already in the explored set, and since our heuristic is consistent, we know that the new path (with B as parent) has a higher cost than the old path (with A as parent), so we can safely ignore the new path.
A* with a **consistent** heuristic

Explored Set
S, B, **A**, C

Frontier
G: $g(G)+h(G)=5$, parent=C

Expand: G
Admissible heuristic example: Romania

Admissible:
\[ h(n) \leq d(n) \]

Example:
\[
\begin{align*}
  d(\text{Sibiu}) &= 278 \\
  h(\text{Sibiu}) &\leq 278
\end{align*}
\]
Consistent heuristic example: Romania

Consistent:
\[ h(n) - h(p) \leq d(n) - d(p) \]

Example:
\[ d(\text{Arad}) - d(\text{Sibiu}) = 140 \]
\[ h(\text{Arad}) - h(\text{Sibiu}) \leq 140 \]

The heuristic difference is always less than or equal to the cost of the action.
Can you use this in the MP?

• Maybe.
• In the MP, every action has a cost of exactly 1!
• ...so a consistent heuristic would be one such that, for every pair of neighboring states n and p, \( h(n) - h(p) \leq 1 \).
• Manhattan distance satisfies this condition.
• There are good heuristics for parts 3 and 4 that don’t satisfy this condition. If your heuristic is not consistent, just make sure that you use an explored dict, instead of an explored set.
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The trivial case: $h(n)=0$

- A heuristic is **admissible** if and only if $d(n) \geq h(n)$ for every $n$.
- A heuristic is **consistent** if and only if $d(n, p) \geq h(n) - h(p)$ for every $n$ and $p$.
- Both criteria are satisfied by $h(n) = 0$. 
UCS = A* with \( h(n) = 0 \)

- Suppose we choose \( h(n) = 0 \)
- Then the frontier is a priority queue sorted by \( g(n) + h(n) = g(n) \)
- In other words, the first node we pull from the queue is the one that’s closest to START!! (The one with minimum \( g(n) \)).
- **Uniform Cost Search** is **A* Search** with the heuristic \( h(n) = 0 \) for all states.
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Heuristics from relaxed problems

• A problem with fewer restrictions on the actions is called a **relaxed problem**

• In most problems, having fewer restrictions on your action means that you can reach the goal faster.

• So designing a heuristic is usually the same as finding a relaxed problem that makes it easy to calculate the distance to goal.
Relaxed heuristic example: Manhattan distance

If there were no walls in the maze, then the number of steps from position \((x_n, y_n)\) to the goal position \((x_G, y_G)\) would be

\[
h(n) = |x_n - x_G| + |y_n - y_G|
\]

If there were no walls, this would be the path to goal: straight down, then straight right.
Relaxed heuristic example: Euclidean distance

If there were no walls in the maze, and we could move diagonally, then the number of steps from position \((x_n, y_n)\) to the goal position \((x_G, y_G)\) would be

\[
h(n) = \sqrt{|x_n - x_G|^2 + |y_n - y_G|^2}
\]
Relaxed heuristic example: Corner dots

Suppose that, instead of touching ALL of the waypoints, you only had to touch the most extreme waypoints?
Relaxed heuristic example: Many dots

Suppose that, after you reached a waypoint, you could magically fly back to the nearest branch in the minimum spanning tree?

In other words, you only have to go one-way from where you are to the waypoint – you don’t have to come back again.
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Which heuristic is better

• If Euclidean distance and Manhattan distance are both admissible heuristics for the single-waypoint maze problem, which one is better?

• Computational complexity of A*: If $c(G)$ is true cost of the best path to goal, then A* evaluates every $n$ for which $g(n) + h(n) \leq c(G)$

• How to minimize computational complexity: make $h(n)$ as large as possible, subject to the constraint that $h(n) \leq d(n)$. 
Euclidean distance

\[ h_2(n) = \sqrt{|x_n - x_G|^2 + |y_n - y_G|^2} \]
Manhattan distance

\[ h_1(n) = |x_n - x_G| + |y_n - y_G| \]

\[ h_1(n) \geq h_2(n) \]

Using \( h_1(n) \), there will be fewer nodes with \( g(n) + h(n) \leq c(G) \). Therefore, computational complexity is lower. Therefore \( h_1(n) \) is better.
Dominance

• If $h_2(n) \geq h_1(n)$ for all $n$, (both admissible) then $h_2$ dominates $h_1$

• As long as they’re both admissible, they will both find the optimum path.

• But $h_2(n)$ will require less computation to find it.
Example: the 8-puzzle

• Problem statement: given a shuffled set of numbers (left), re-arrange them in order (right).

• State: ordering of the numbers and of the space.

• Possible actions: swap the space with any of its neighbors.

• Like traveling salesman, this is an NP-complete problem.
8-puzzle: Heuristic $h_1(n)$

• Suppose that, on each step, we could move any tile, anywhere on the board, regardless of where other tiles were.
• Then $h_1(n) = \# \text{tiles that need to be moved}.$
• Example below: $h_1(n) = 8$

![Start State](image1)

![Goal State](image2)
8-puzzle: Heuristic $h_2(n)$

• Suppose that, on each step, we could move any tile by just one step horizontally or vertically, regardless of whether there are other tiles in the way.

• Then $h_2(n) = \text{sum of Manhattan distances from the current positions of each tile to their target positions}$ (notice: $h_2(n) \geq h_1(n)$)

• Example below: $h_2(n) = 3 + 1 + 2 + 2 + 2 + 3 + 3 + 2 = 18$
Dominance
Experiment results reported by Svetlana Lazebnik

Typical search costs for the 8-puzzle (average number of nodes expanded for different solution depths):

- $d=12$
  - BFS expands 3,644,035 nodes
  - $A^*(h_1)$ expands 227 nodes
  - $A^*(h_2)$ expands 73 nodes

- $d=24$
  - BFS expands 54,000,000,000 nodes
  - $A^*(h_1)$ expands 39,135 nodes
  - $A^*(h_2)$ expands 1,641 nodes
Combining heuristics

• Suppose we have a collection of admissible heuristics $h_1(n)$, $h_2(n)$, ..., $h_m(n)$, but none of them dominates the others
• How can we combine them?

$$h(n) = \max\{h_1(n), h_2(n), ..., h_m(n)\}$$
Outline of lecture

1. Admissible heuristics: $h(n) \leq d(n)$
2. Consistent heuristics: $h(n) - h(p) \leq d(n) - d(p)$
3. The zero heuristic: Uniform Cost Search: $h(n) = 0$
4. Relaxed heuristics: $h(n)$ is the $d(n)$ from a problem with fewer rules.
5. Dominant heuristics: if $h_2(n) \leq h_1(n)$ and both are admissible, then $h_1(n)$ has lower computational complexity
# Five search strategies

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Complete?</th>
<th>Optimal?</th>
<th>Time complexity</th>
<th>Space complexity</th>
<th>Implement the Frontier as a...</th>
</tr>
</thead>
<tbody>
<tr>
<td>BFS</td>
<td>Yes</td>
<td>If all step costs equal</td>
<td>$O{b^d}$</td>
<td>$O{b^d}$</td>
<td>Queue</td>
</tr>
<tr>
<td>DFS</td>
<td>No</td>
<td>No</td>
<td>$O{b^m}$</td>
<td>$O{bm}$</td>
<td>Stack</td>
</tr>
<tr>
<td>UCS</td>
<td>Yes</td>
<td>Yes</td>
<td>#nodes s.t. $g(n) \leq c(G)$</td>
<td>#nodes s.t. $g(n) \leq c(G)$</td>
<td>Priority Queue: $g(n)$</td>
</tr>
<tr>
<td>Greedy</td>
<td>No</td>
<td>No</td>
<td>$O{b^m}$</td>
<td>$O{b^m}$</td>
<td>Priority Queue: $h(n)$</td>
</tr>
<tr>
<td>A*</td>
<td>Yes</td>
<td>Yes</td>
<td>#nodes s.t. $g(n) + h(n) \leq c(G)$</td>
<td>#nodes s.t. $g(n) + h(n) \leq c(G)$</td>
<td>Priority Queue: $h(n) + g(n)$</td>
</tr>
</tbody>
</table>