CS440/ECE448 Lecture 22: Markov Decision Processes

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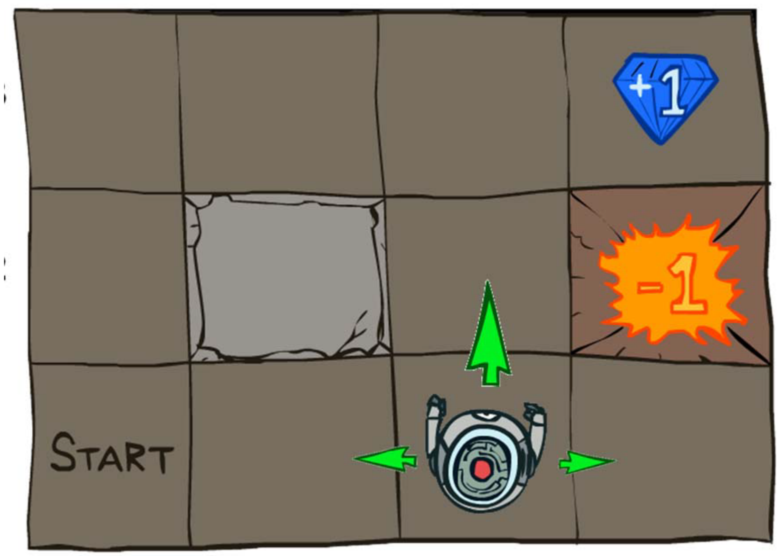
Including slides by Svetlana Lazebnik, 11/2016

Including many figures by Peter Abbeel and Dan Klein, UC Berkeley CS 188



Grid World

Invented and drawn by Peter Abbeel and Dan Klein, UC Berkeley CS 188

Markov Decision Processes

- In HMMs, we see a sequence of observations and try to reason about the underlying state sequence
  - There are no actions involved
- But what if we have to take an action at each step that, in turn, will affect the state of the world?

Markov Decision Processes

Components that define the MDP. Depending on the problem statement, you either know these, or you learn them from data:

- Like an HMM, we have **States** s, beginning with initial state s_0
- Unlike an HMM, we also have **Actions** a
  - Each state s has actions A(s) available from it
- Unlike an HMM, the **Transition model** P(s' | s, a) depends on both the state you're in, and the action you perform.
  - *Markov assumption*: the probability of going to s' from s depends only on s and a and not on any other past actions or states

Markov Decision Processes

What is our purpose? In an HMM, we're trying to understand the world. In an MDP, we're trying to influence the world. Why?

- **Reward function** $R(s)$. As in a two-player game, we assume that some states are better than others. Our goal is to influence the world so that we end up in a better state.

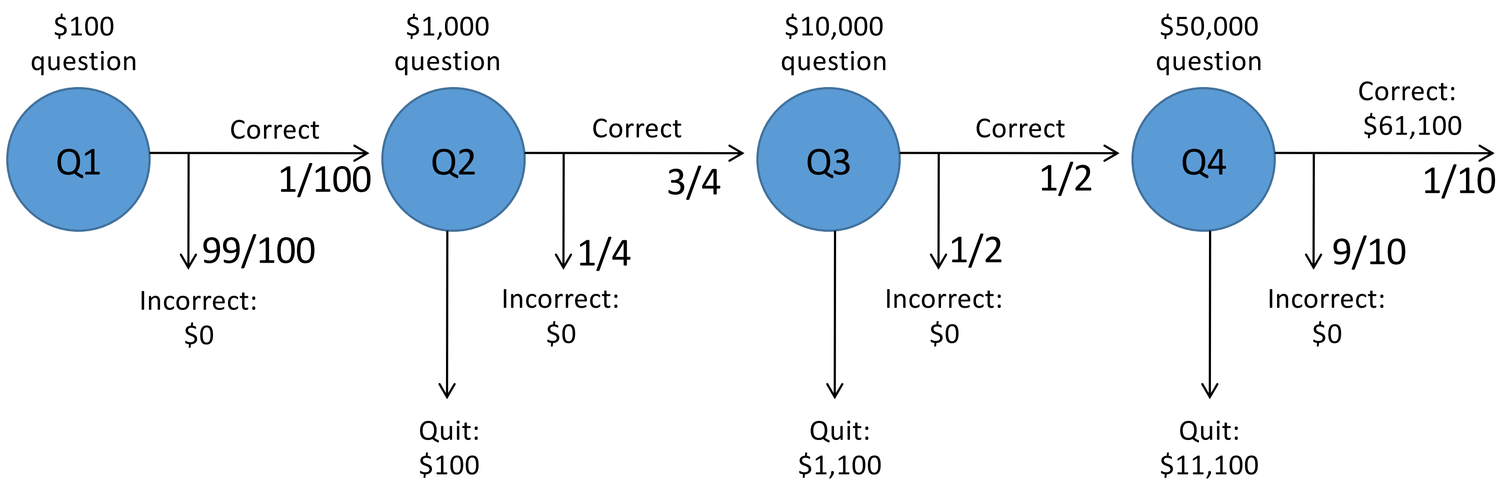
- **The solution: A Policy,** $\pi(s) \in A(s)$: the action that the agent takes in any given state**.** As in a two-player game, we are trying to solve for a policy that is optimum, in the sense that it maximizes our expected reward.

Outline

1. What we need to know: transition model, reward function.
2. The closed-form optimum solution: The Bellman equation.
3. Iterative solutions that approximate the Bellman equation in polynomial time
    1. Value Iteration
    2. Policy Iteration

Example: Game show

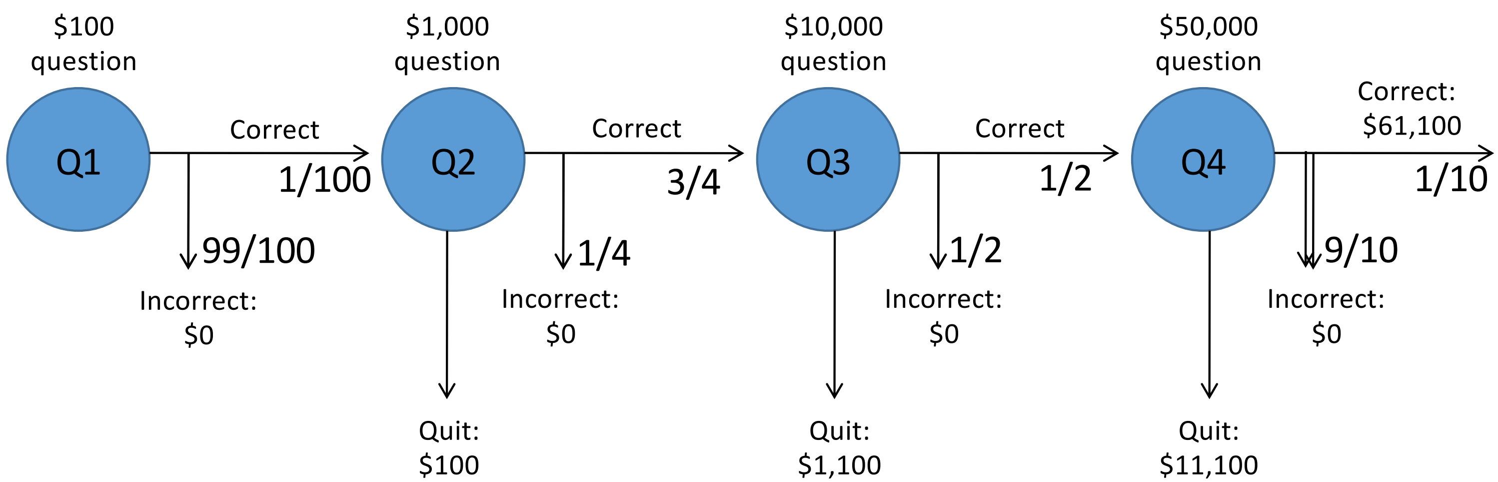
- A series of questions with increasing level of difficulty and increasing payoff
- Decision: at each step, take your earnings and quit, or go for the next question
  - If you answer wrong, you lose everything



Example: Game show

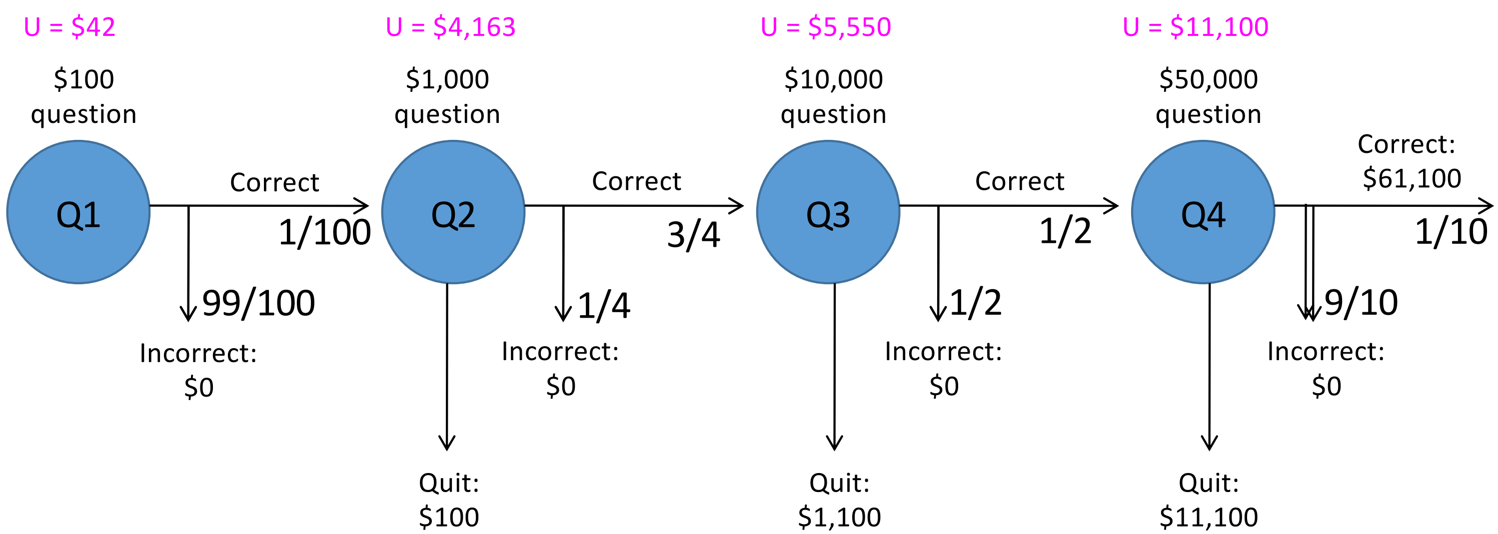
- Consider $50,000 question
  - Probability of guessing correctly: 1/10
  - Quit or go for the question?
- What is the expected payoff for continuing?

  0.1 * 61,100 + 0.9 * 0 = 6,110
- What is the optimal decision?

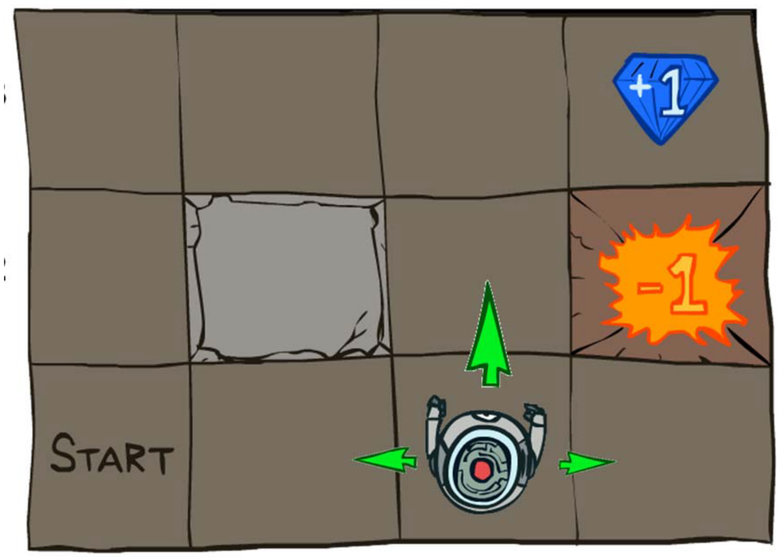


Example: Game show

- What should we do in Q3?
  - Payoff for quitting: $1,100
  - Payoff for continuing: 0.5 * $11,100 = $5,550
- What about Q2?
  - $100 for quitting vs. $4,162 for continuing
- What about Q1?

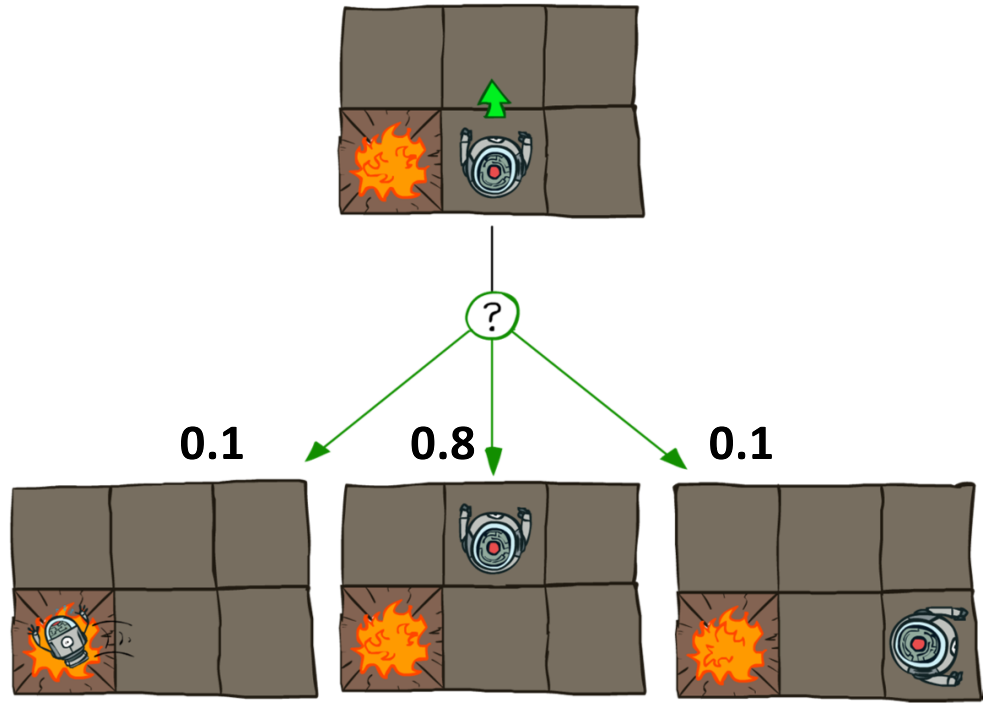


Example: Grid world



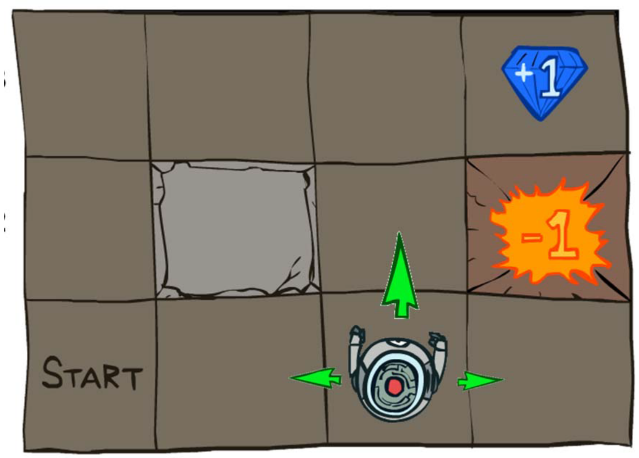
R(s) = -0.04 for every non-terminal state

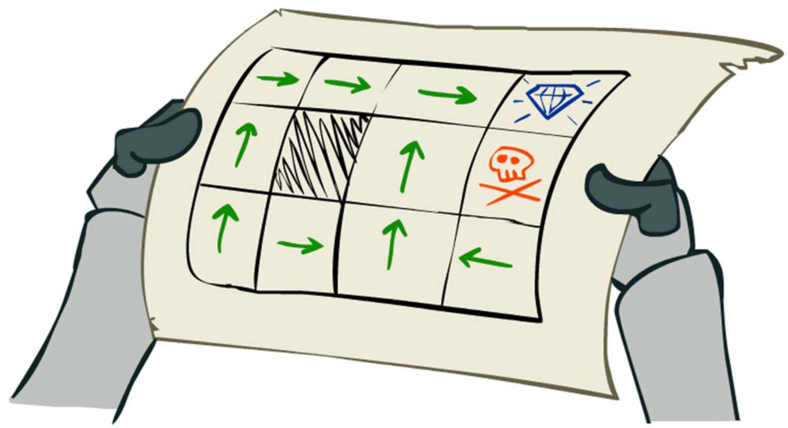
Transition model:



Source: P. Abbeel and D. Klein

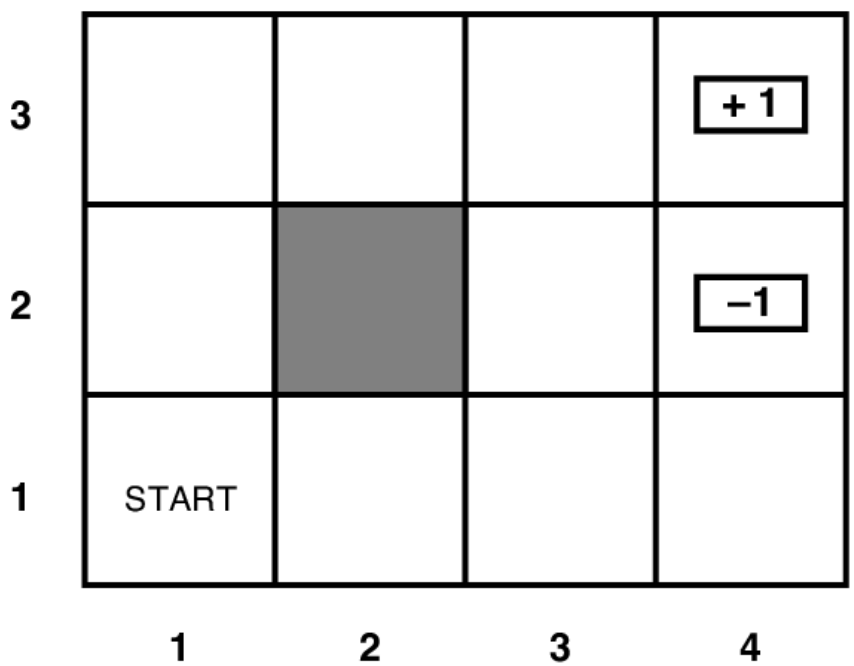
Goal: Policy



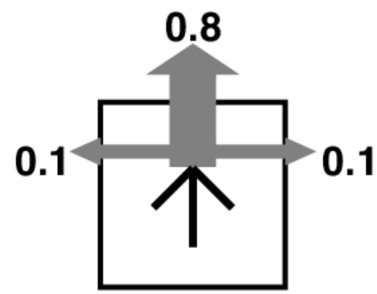


Source: P. Abbeel and D. Klein

Grid world

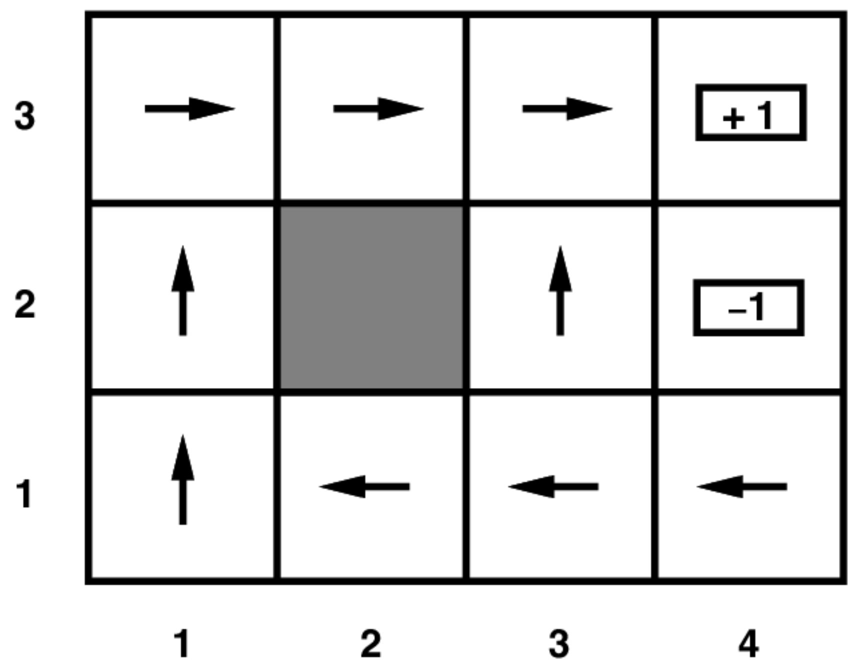


Transition model:



R(s) = -0.04 for every non-terminal state

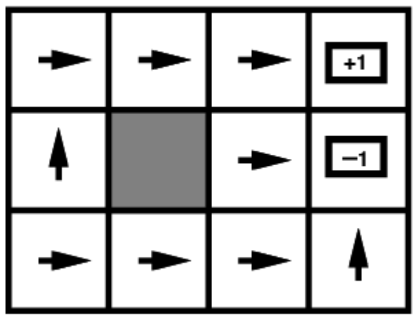
Grid world



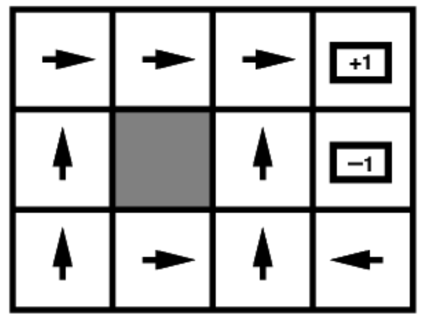
Optimal policy when R(s) = -0.04 for every non-terminal state

Grid world

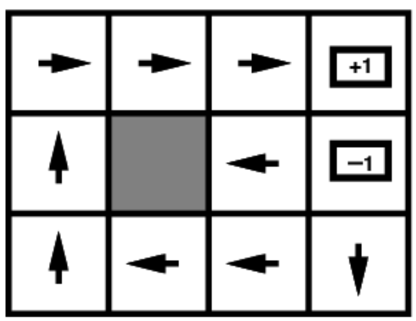
- Optimal policies for other values of R(s):



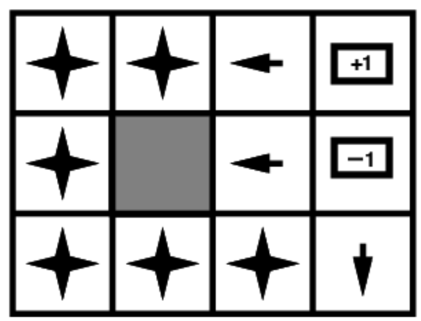
$R(s) < -1.6284$



$-0.4278 < R(s) < -0.0850$



$-0.0221 < R(s) < 0$



$R(s) > 0$

Solving MDPs

- MDP components:
  - **States** s
  - **Actions** a
  - **Transition model** $P(s' \mid s, a)$
  - **Reward function** $R(s)$
- The solution:
  - **Policy** $\pi(s)$: mapping from states to actions
  - How to find the optimal policy?

Outline

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Maximizing expected utility

- The optimal policy π(s) should maximize the *expected utility* over all possible state sequences produced by following that policy:

$$\sum_{\substack{state\ sequences \\ starting\ from\ s_0}} P\big(sequence \mid s_0, a = \pi(s_0)\big)\, U(sequence)$$

- How to define the utility of a state sequence?
  - Sum of rewards of individual states
  - Problem: infinite state sequences

Utilities of state sequences

- Normally, we would define the utility of a state sequence as the sum of the rewards of the individual states
- **Problem:** infinite state sequences
- **Solution:** *discount* the individual state rewards by a factor γ between 0 and 1:

$$U([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots$$

$$= \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \frac{R_{\max}}{1-\gamma} \qquad (0 < \gamma < 1)$$

  - Sooner rewards count more than later rewards
  - Makes sure the total utility stays bounded
  - Helps algorithms converge

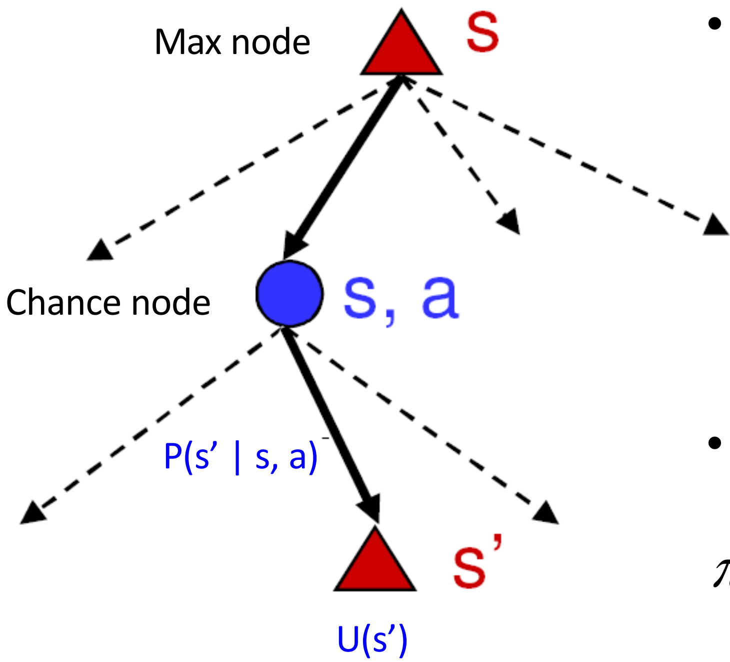
Utilities of states

- Expected utility obtained by policy π starting in state s:

$$U^{\pi}(s) = \sum_{\substack{state\ sequences \\ starting\ from\ s}} P\big(sequence | s, a = \pi(s)\big) U(sequence)$$

- The "true" utility of a state, denoted U(s), is the *best possible* expected sum of discounted rewards
  - if the agent executes the *best possible* policy starting in state s
- Reminiscent of minimax values of states…

Finding the utilities of states



- If state s' has utility U(s'), then what is the expected utility of taking action **a** in state **s**?

$$\sum_{s'} P(s' | s, a) U(s')$$

- How do we choose the optimal action?

$$\pi^*(s) = \arg\max_{a \in A(s)} \sum_{s'} P(s' | s, a) U(s')$$

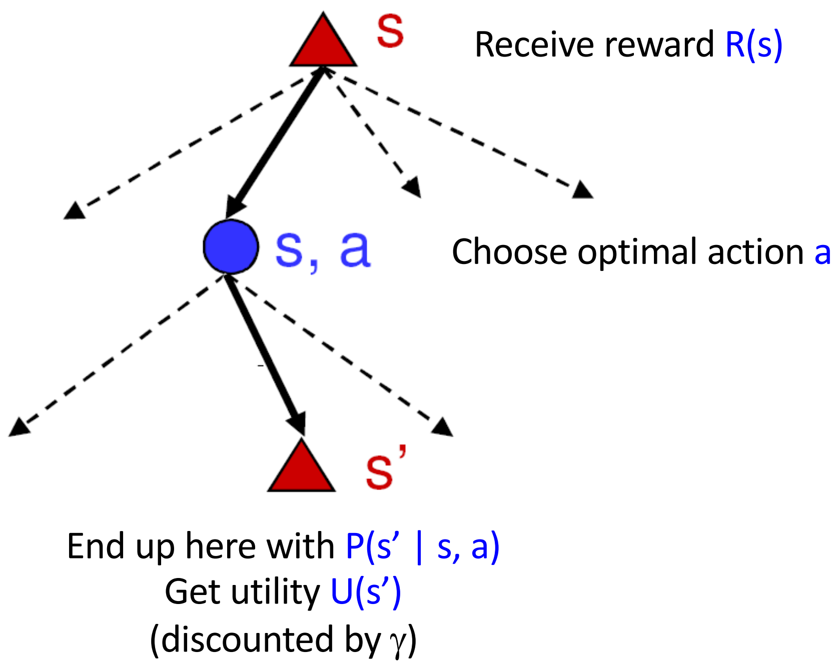
- What is the recursive expression for **U(s)** in terms of the utilities of its successor states?

$$U(s) = R(s) + \gamma \max_a \sum_{s'} P(s' | s, a) U(s')$$

The Bellman equation

- Recursive relationship between the utilities of successive states:

$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')$$



The Bellman equation

- Recursive relationship between the utilities of successive states:

$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' | s, a) U(s')$$

- For *N* states, we get *N* **nonlinear** equations in *N* unknowns
  - Known quantities: $P(s'|s,a)$, $R(s)$, and γ. Unknowns: $U(s)$.
  - Solving these N equations solves the MDP.
  - Nonlinear -> no closed-form solution.
    - If it weren't for the "max," this would be N linear equations in N unknowns. We could solve it by just inverting an NxN matrix.
    - The "max" means that there is no closed-form solution. Need to use an iterative solution method, which might not converge to the globally optimum soluton.
  - Two solution methods: **value iteration** and **policy iteration**

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Method 1: Value iteration

- Start out with iteration $i = 0$, every $U_i\ (s)\ =\ 0$
- Iterate until convergence
  - During the i^{th} iteration, update the utility of each state according to this rule:

$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U_i(s')$$

- In the limit of infinitely many iterations, guaranteed to find the correct utility values.
  - Error decreases exponentially, so in practice, don't need an infinite number of iterations…

Value Iteration: Iteration 1

$$U_1(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) U_0(s')$$

$U_1(s)$

-0.04	-0.04	-0.04	
-0.04		-0.04	
-0.04	-0.04	-0.04	-0.04

$R(s)$

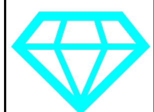
-0.04	-0.04	-0.04	
-0.04		-0.04	
-0.04	-0.04	-0.04	-0.04

$U_0(s)$

0	0	0	
0		0	
0	0	0	0

Value Iteration: Iteration 2

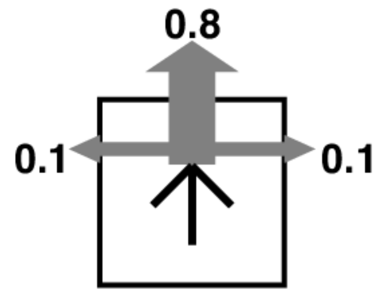
$U_1(s)$

-0.04	-0.04	-0.04	
-0.04		-0.04	
-0.04	-0.04	-0.04	-0.04

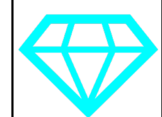
Value Iteration: Iteration 2

$$U_2(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) U_1(s')$$

Transition model:



$U_1(s)$

-0.04	-0.04	-0.04	
-0.04		-0.04	
-0.04	-0.04	-0.04	-0.04

$$P(s'|s,\text{up}) = \begin{cases} 0.8 & s' = \text{up from } s \text{ (if no wall)} \\ 0.1 & s' = \text{left from } s \text{ (if no wall)} \\ 0.1 & s' = \text{right from } s \text{ (if no wall)} \end{cases}$$

Value Iteration: Iteration 2

$$U_2(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a)U_1(s')$$

$U_2(s)\ (\gamma = 1)$

−0.08	−0.08	+0.75	
−0.08		−0.08	
−0.08	−0.08	−0.08	−0.08

$\sum_{s'} P(s'|s,\text{down})U_1(s)$

−0.04	−0.04	+0.06	
−0.04		−0.14	
−0.04	−0.04	−0.04	−0.04

$\sum_{s'} P(s'|s,\text{up})U_1(s)$

−0.04	−0.04	+0.06	
−0.04		−0.14	
−0.04	−0.04	−0.04	−0.81

$\sum_{s'} P(s'|s,\text{left})U_1(s)$

−0.04	−0.04	−0.04	
−0.04		−0.04	
−0.04	−0.04	−0.04	−0.14

$\sum_{s'} P(s'|s,\text{right})U_1(s)$

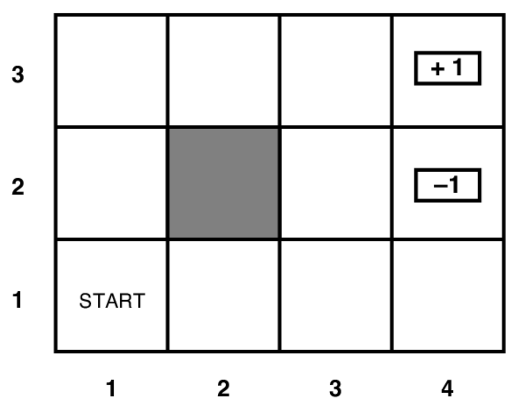
−0.04	−0.04	+0.79	
−0.04		−0.81	
−0.04	−0.04	−0.04	−0.14

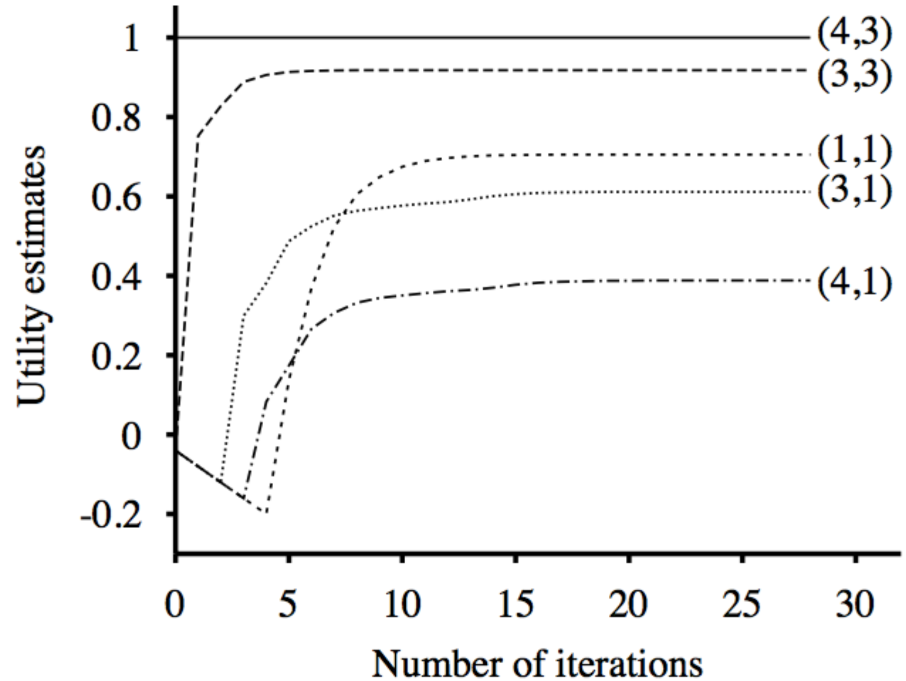
$U_1(s)$

−0.04	−0.04	−0.04	
−0.04		−0.04	
−0.04	−0.04	−0.04	−0.04

Value iteration

Input (non-terminal R=-0.04)

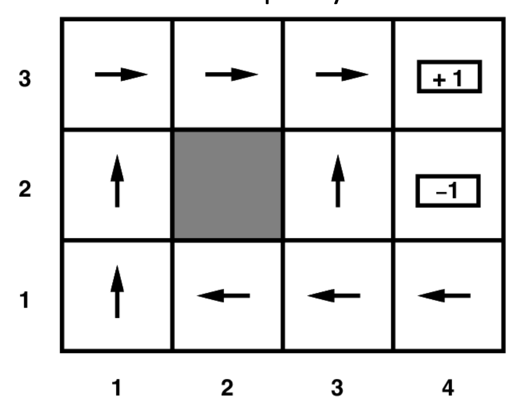




Utilities with discount factor 1

	1	2	3	4
3	0.812	0.868	0.918	+1
2	0.762		0.660	–1
1	0.705	0.655	0.611	0.388

Final policy



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Method 2: Policy Iteration

- Start with some initial policy π_0 and alternate between the following steps:
  - **Policy Evaluation:** calculate the utility of every state under the assumption that the given policy is fixed and unchanging.
  - **Policy Improvement:** calculate a new policy π_{i+1} based on the updated utilities.

- Notice it's kind of like gradient descent in neural networks:
  - Policy evaluation: Find ways in which the current policy is suboptimal
  - Policy improvement: Fix those problems

- Unlike Value Iteration, this is guaranteed to converge in a finite number of steps, as long as the state space and action set are both finite.

Method 2: Policy Iteration

Policy Evaluation: Given a fixed policy π, calculate the **policy-dependent utility**, $U^\pi(s)$, for every state *s*

$$U^\pi(s) = R(s) + \gamma \sum_{s'} P(s'|s,\pi(s)) U^\pi(s')$$

Notice how this differs from the Bellman equation:

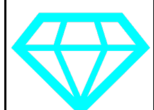
$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) U(s')$$

The difference is that policy evaluation is N _linear_ equations in N unknowns, whereas the Bellman equation is N _nonlinear_ equations in N unknowns (N=# states).

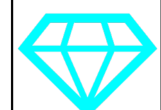
Policy Iteration: Iteration 1

Policy Evaluation: $U^{\pi^0}(s) = R(s) + \gamma \sum_{s'} P(s'|s,a) U^{\pi^0}(s')$

$U^{\pi^0}(s)$

$+0.50$	$+0.69$	$+0.74$	
-0.65		-0.90	
-1.40	-1.44	-1.39	-1.40

$\pi^0(s)$

$\rightarrow$	$\rightarrow$	$\rightarrow$	
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$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$

Method 2: Policy Iteration

- **Policy Evaluation**: Given a fixed policy π, calculate the **policy-dependent utility**, $U^\pi(s)$, for every state *s*

$$U^\pi(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi(s)) U^\pi(s')$$

- **Policy Improvement**: Given $U^\pi(s)$ for every state *s*, find an improved π(s)

$$\pi^{i+1}(s) = \arg\max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U^{\pi_i}(s')$$

Policy Iteration: Iteration 1

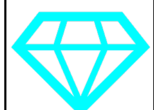
Policy Evaluation: $U^{\pi^0}(s) = R(s) + \gamma \sum_{s'} P(s'|s,a)U^{\pi^0}(s')$

Policy Improvement: $\pi^1(s) = \underset{a}{\text{argmax}} \sum_{s'} P(s'|s,a)U^{\pi^0}(s')$

$\pi^1(s)$

$\rightarrow$	$\rightarrow$	$\rightarrow$	
$\uparrow$		$\uparrow$	
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$

$U^{\pi^0}(s)$

$+0.50$	$+0.69$	$+0.74$	
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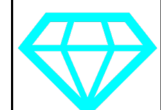
$\pi^0(s)$

$\rightarrow$	$\rightarrow$	$\rightarrow$	
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Summary

- MDP defined by states, actions, transition model, reward function
- The "solution" to an MDP is the policy: what do you do when you're in any given state
- The Bellman equation tells the utility of any given state, and incidentally, also tells you the optimum policy. The Bellman equation is N nonlinear equations in N unknowns (the policy), therefore it can't be solved in closed form.
- Value iteration:
  - At the beginning of the (i+1)'st iteration, each state's value is based on looking ahead i steps in time
  - … so finding the best action = optimize based on (i+1)-step lookahead
- Policy iteration:
  - Find the utilities that result from the current policy,
  - Improve the current policy