Lecture 3: Uniform Cost Search and A*

Mark Hasegawa-Johnson, January 2021

With some slides by Svetlana Lazebnik, 9/2016

Review: DFS and BFS

- Breadth-first search
  - Frontier is a queue: expand the shallowest node
  - **Complete**: always finds a solution, if one exists
  - **Optimal** (finds the best solution) **if all actions have the same cost**.
  - **Time complexity:** $O\{b^d\}$
  - **Space complexity:** $O\{b^d\}$.
- Depth-first search – utility depends on relationship between *m* and *d*
  - Frontier is a stack: expand the deepest node
  - **Not complete** (might never find a solution, if *m* is infinite)
  - **Not optimal** (returned solution is rarely the best one)
  - **Time complexity:** $O\{b^m\}$
  - **Space complexity:** $O\{bm\}$.

Outline of today's lecture

1. Uniform Cost Search (UCS): like BFS, but for actions that have different costs
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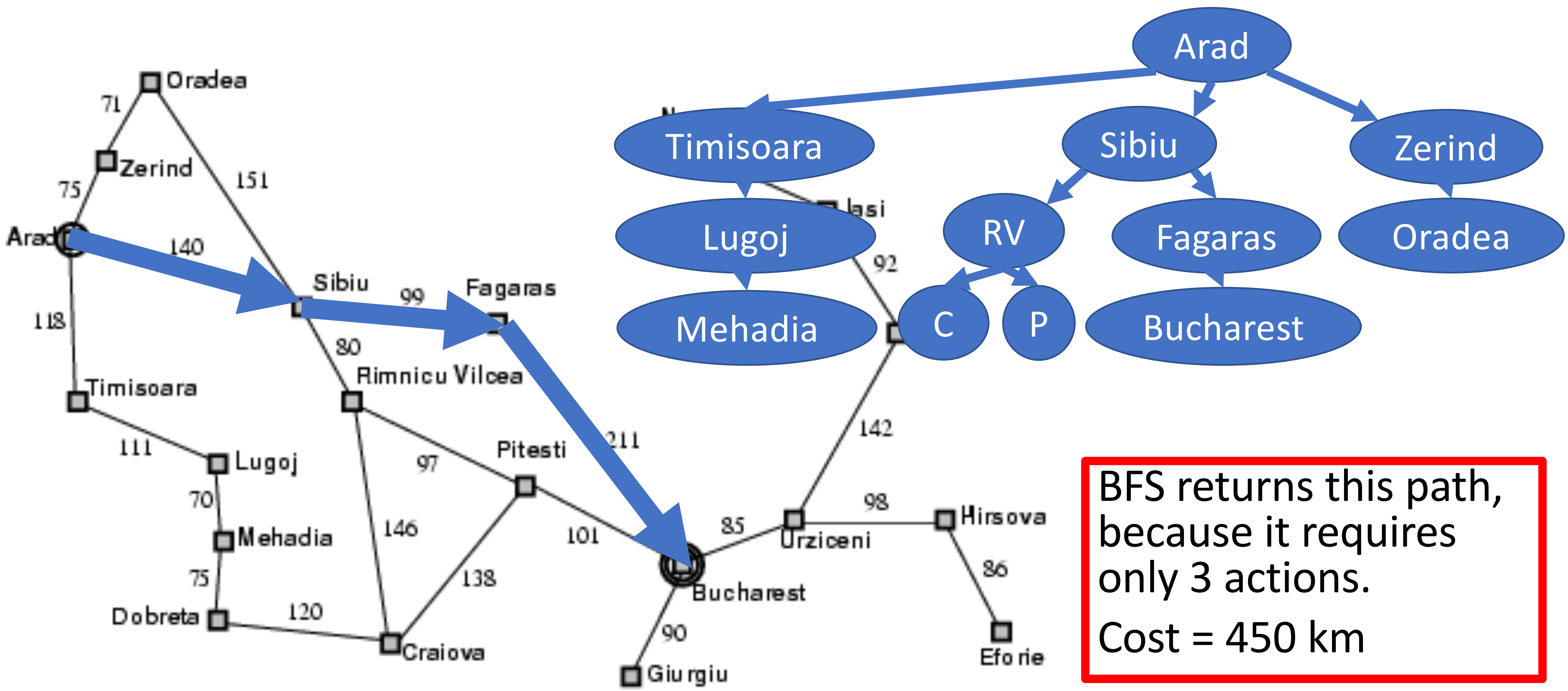
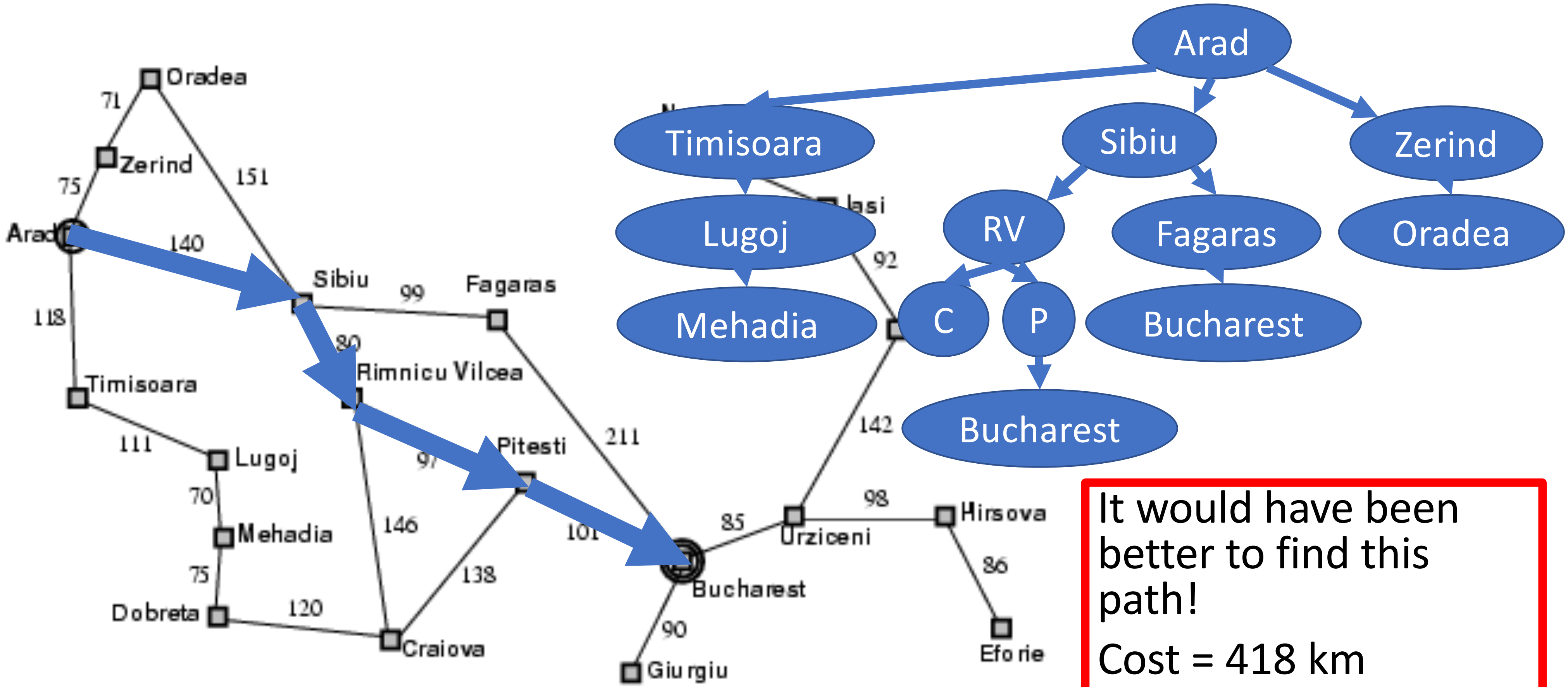
An example for which BFS is not optimal: Romania



BFS returns this path, because it requires only 3 actions.

Cost = 450 km

An example for which BFS is not optimal: Romania



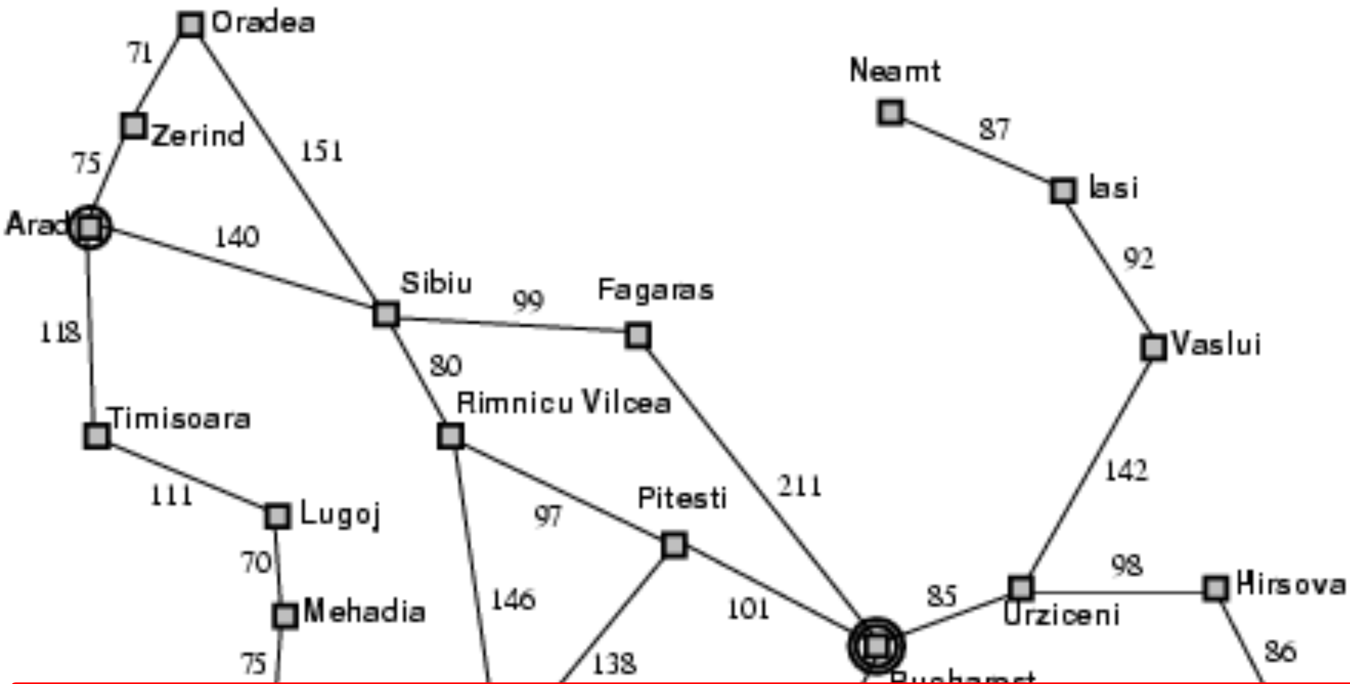
It would have been better to find this path!

Cost = 418 km

The solution: Uniform Cost Search

- Breadth-first search (BFS): Next node expanded is the one with the fewest required actions
  - Frontier is a queue
  - First node into the queue is the first one expanded (FIFO)
- Uniform cost search (UCS): Next node expanded is the one with the lowest accumulated path cost
  - Frontier is a ***priority queue***
  - ***Lowest-cost node*** is the first one expanded

Example of UCS: Romania



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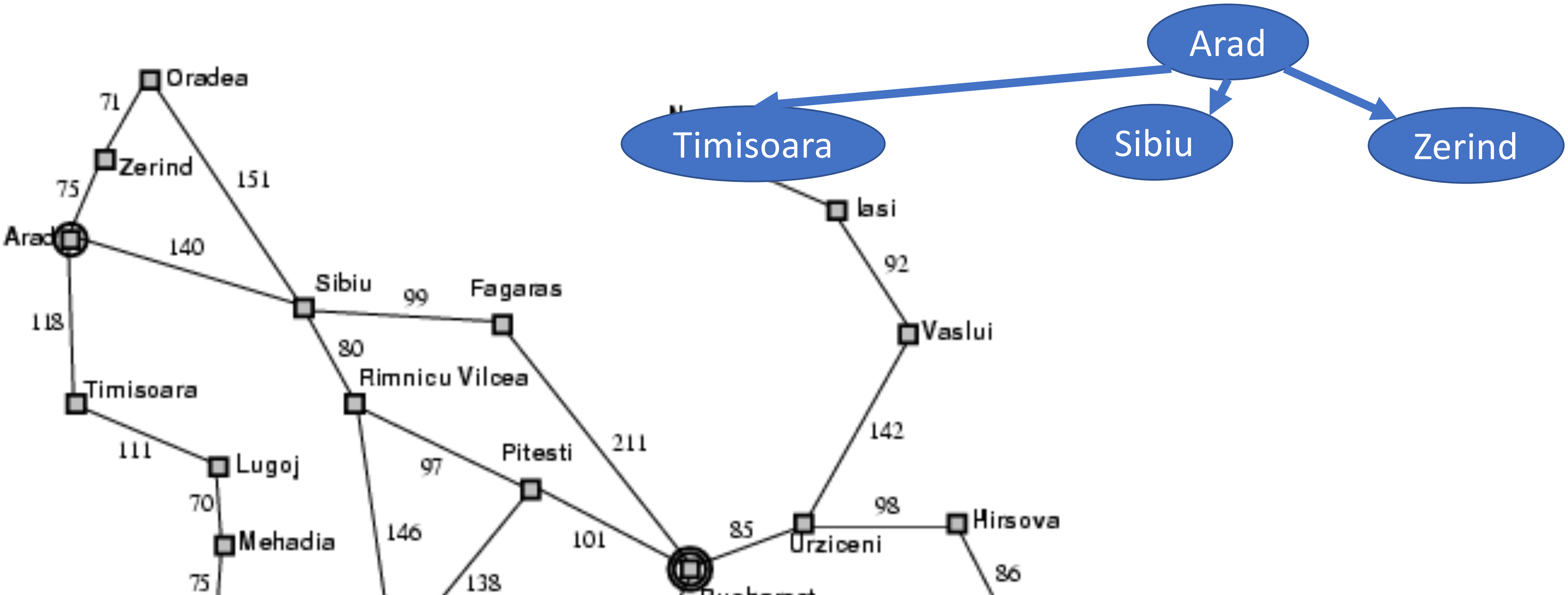
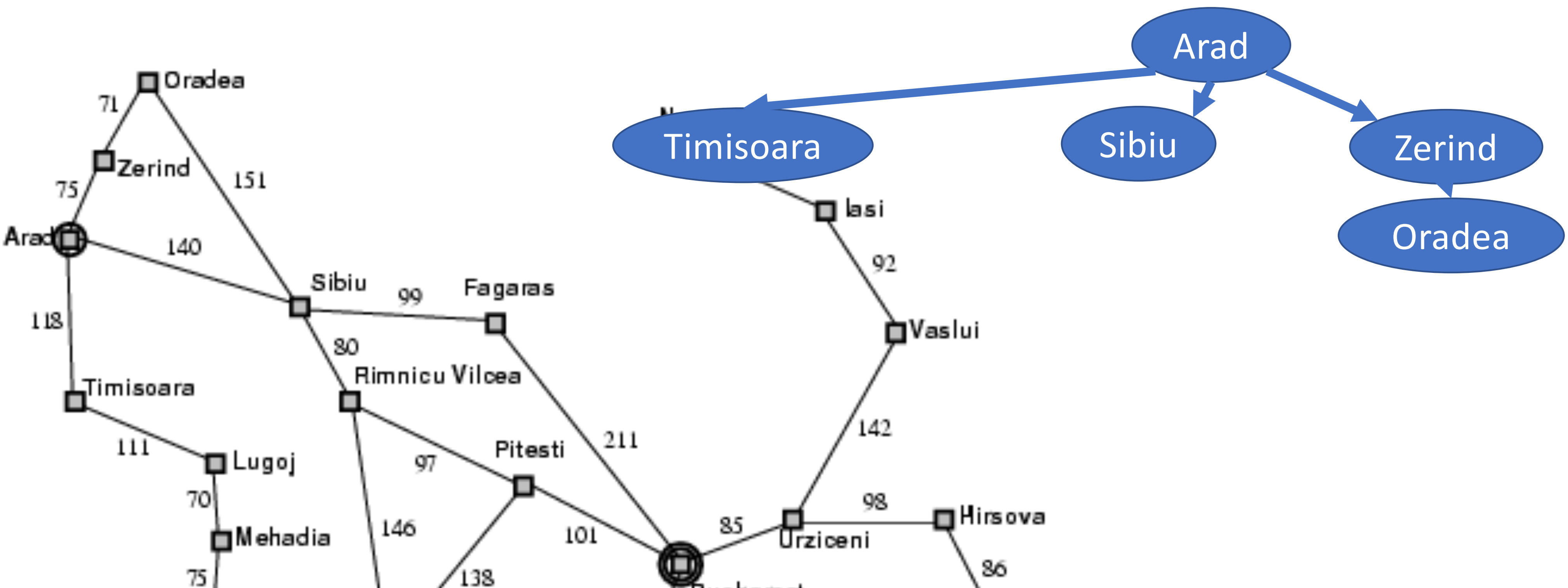
Example of UCS: Romania



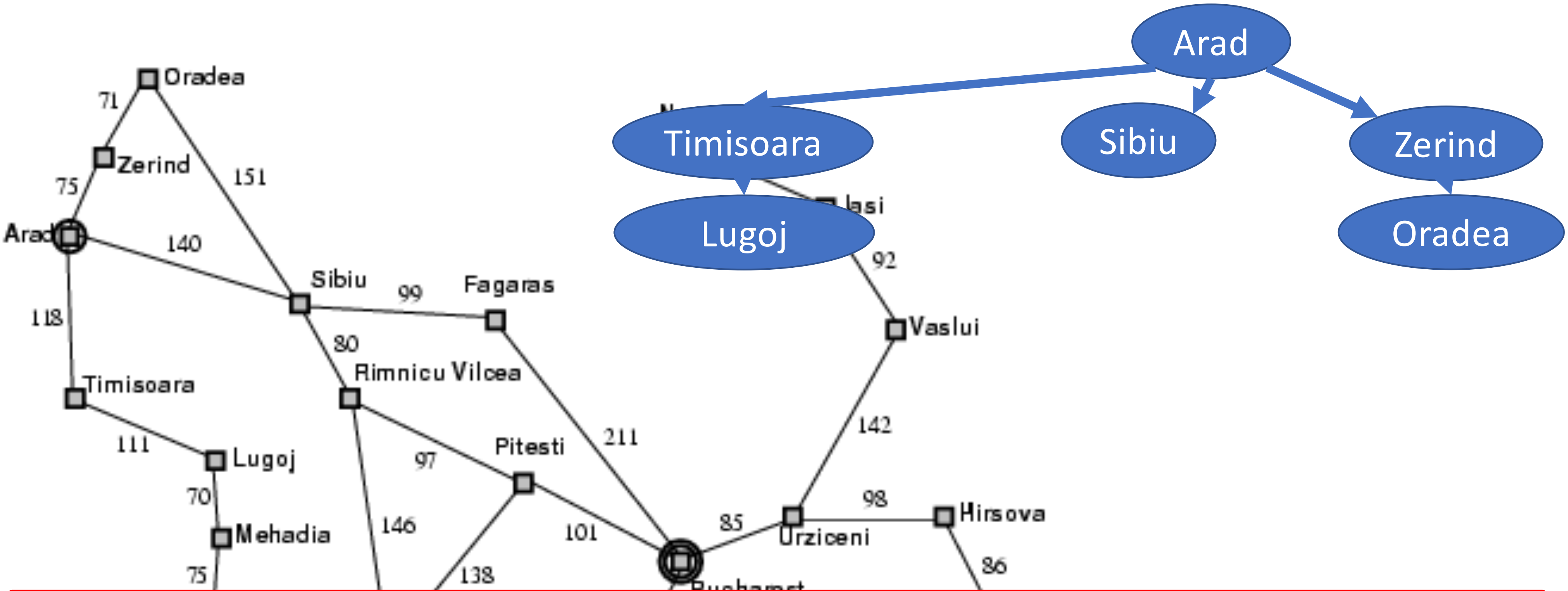
<u>Zerind:75, Timisoara:118, Sibiu:140</u>

Example of UCS: Romania



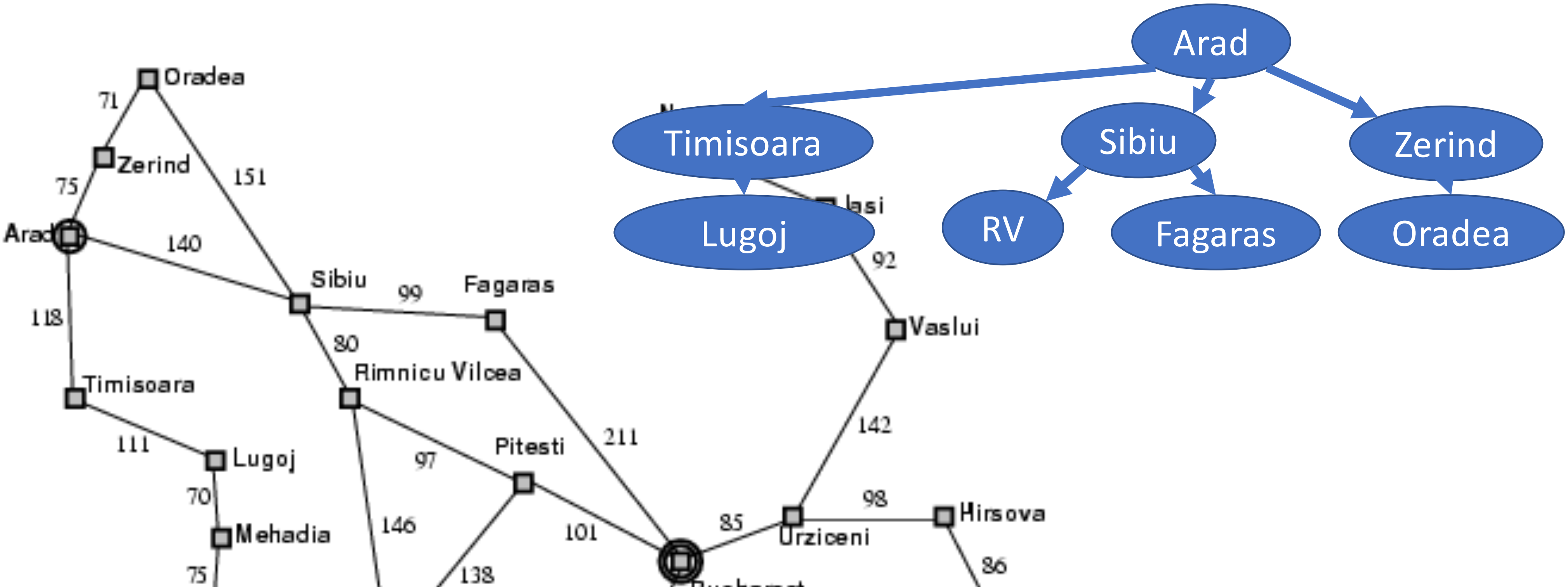
Timisoara:118, Sibiu:140, **<u>Oradea:146</u>**

Example of UCS: Romania



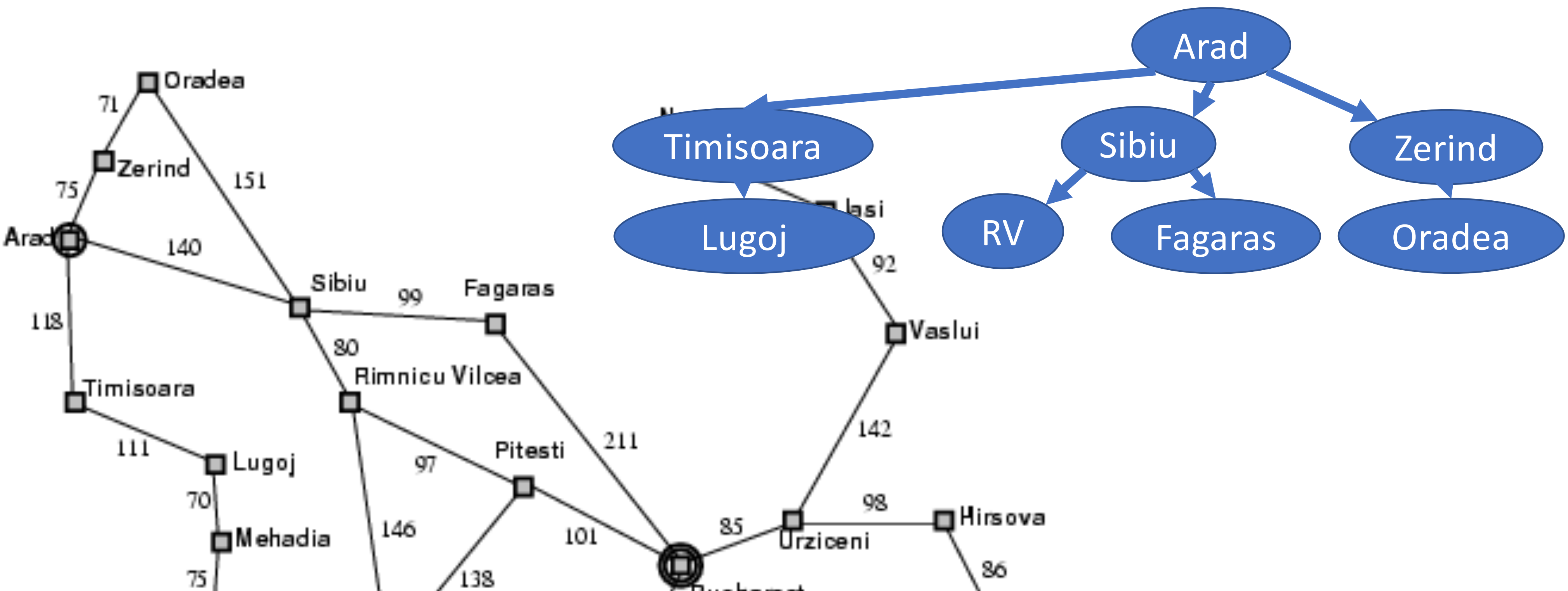
Sibiu:140, Oradea:146, **<u>Lugoj:239</u>**

Example of UCS: Romania



Oradea:146, **Rimnicu Vilcea:220**, Lugoj:239, **Fagaras:239**

Example of UCS: Romania



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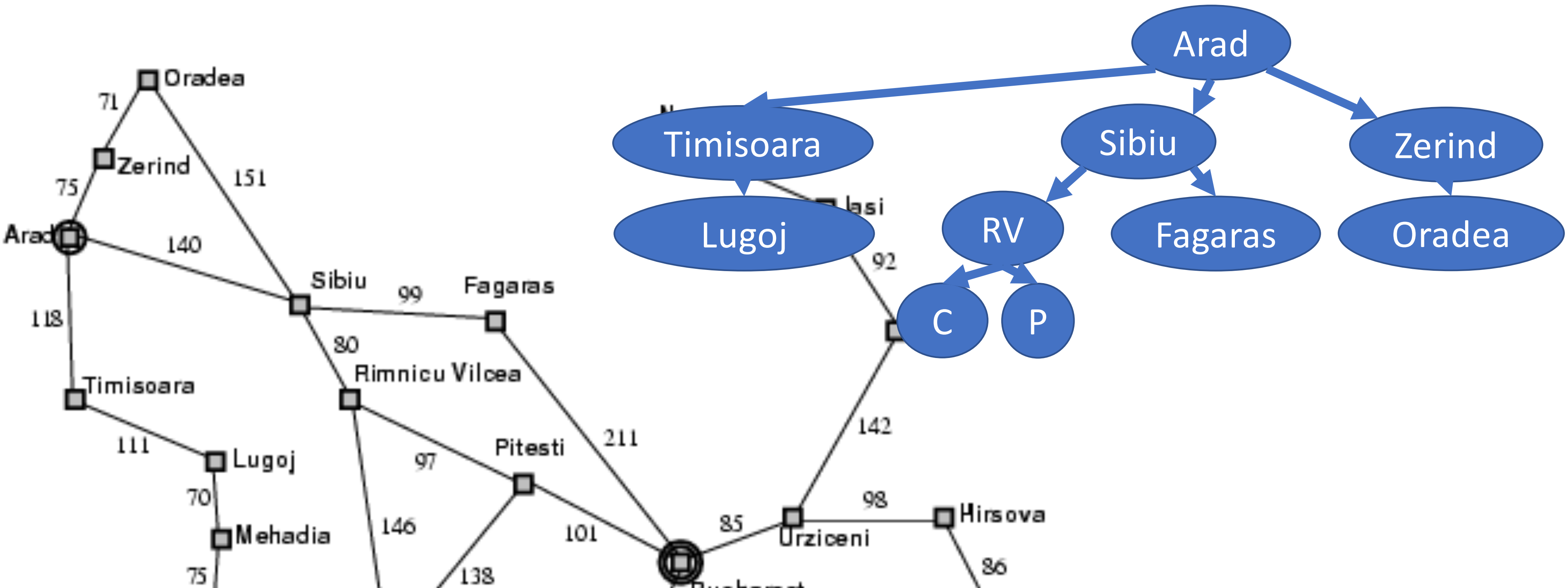
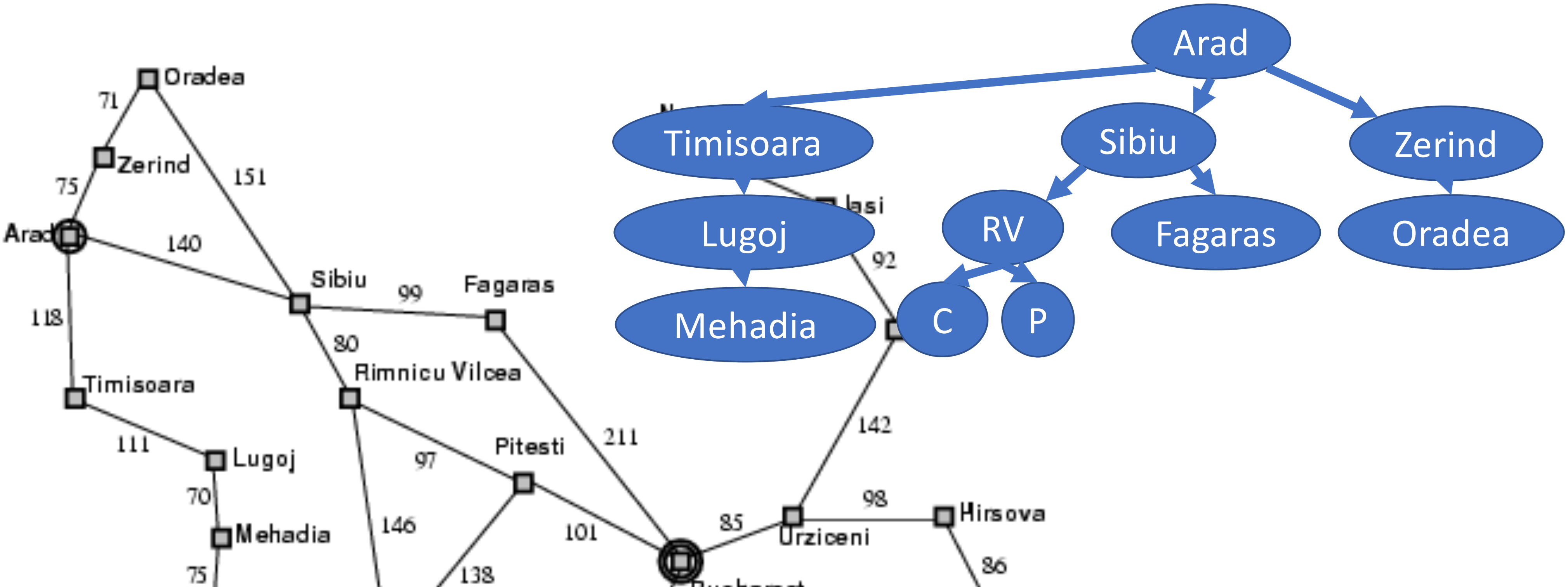
Example of UCS: Romania



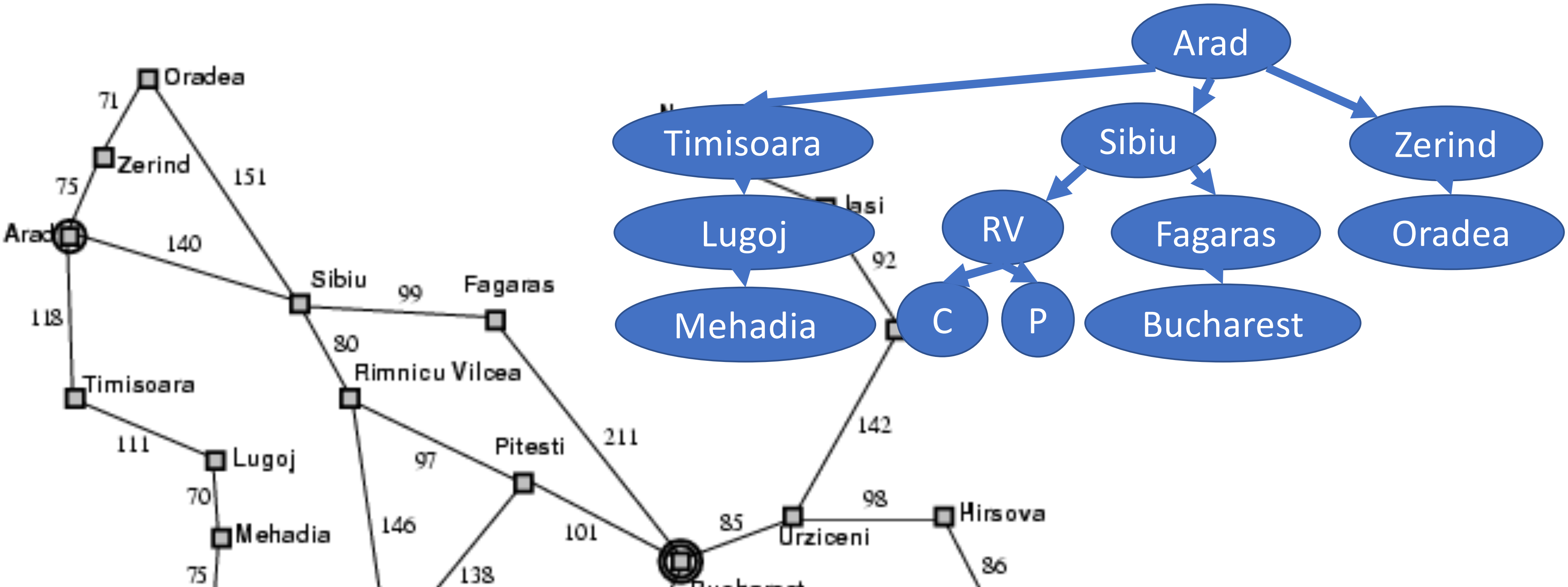
Lugoj:239, Fagaras:239, **Pitesti:317, Craiova:366**

Example of UCS: Romania



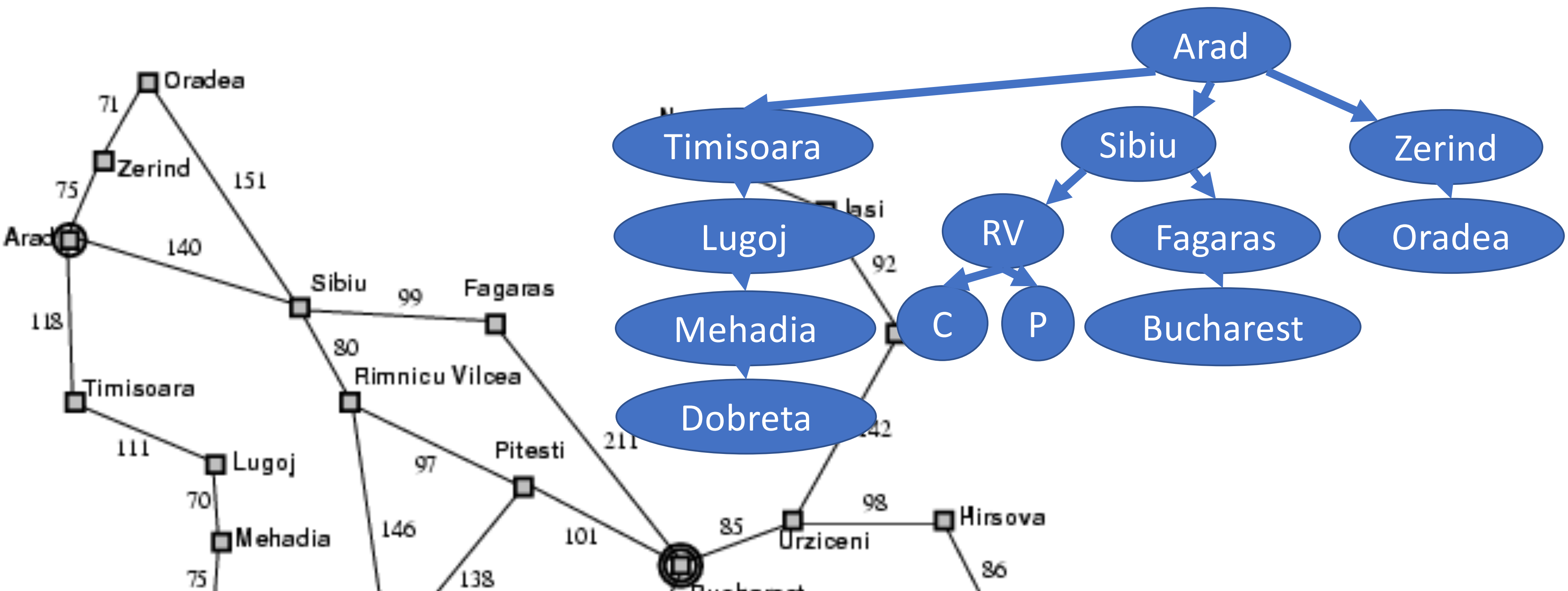
Fagaras:239, **<u>Mehadia:309</u>**, Pitesti:317, Craiova:366

Example of UCS: Romania



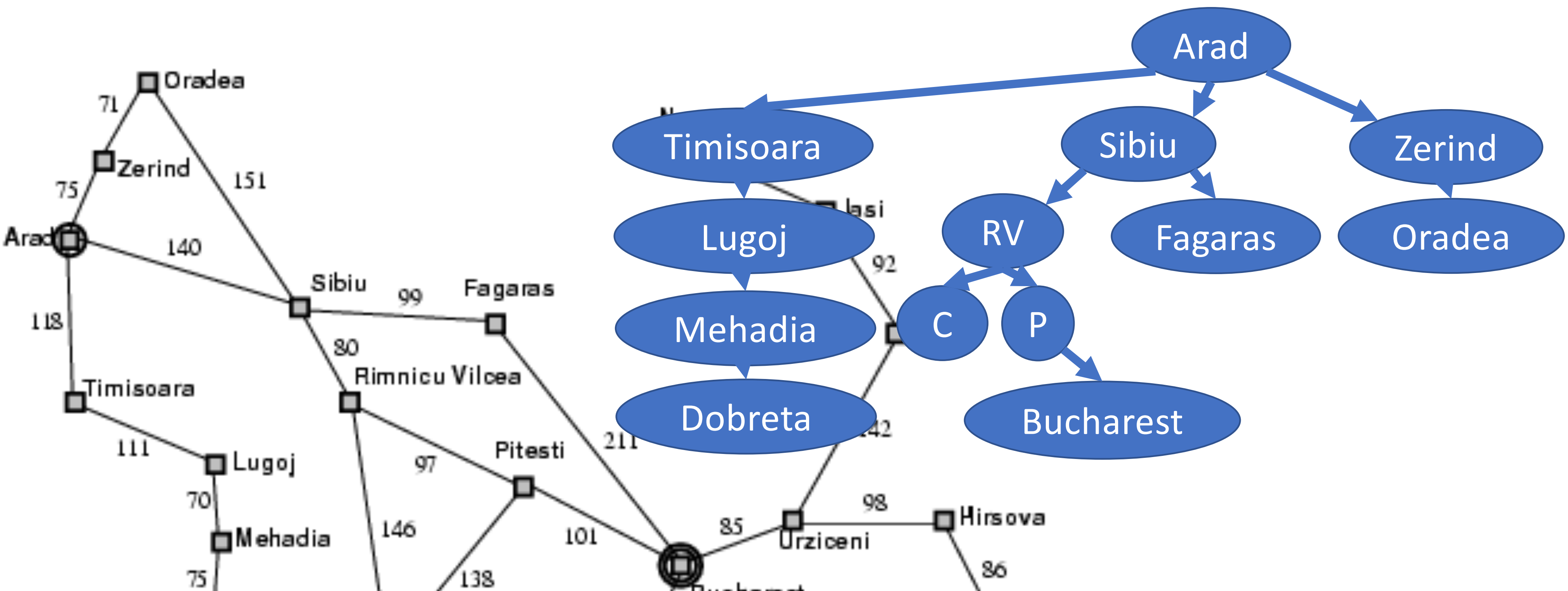
Mehadia:309, Pitesti:317, Craiova:366, **Bucharest:450**

Example of UCS: Romania



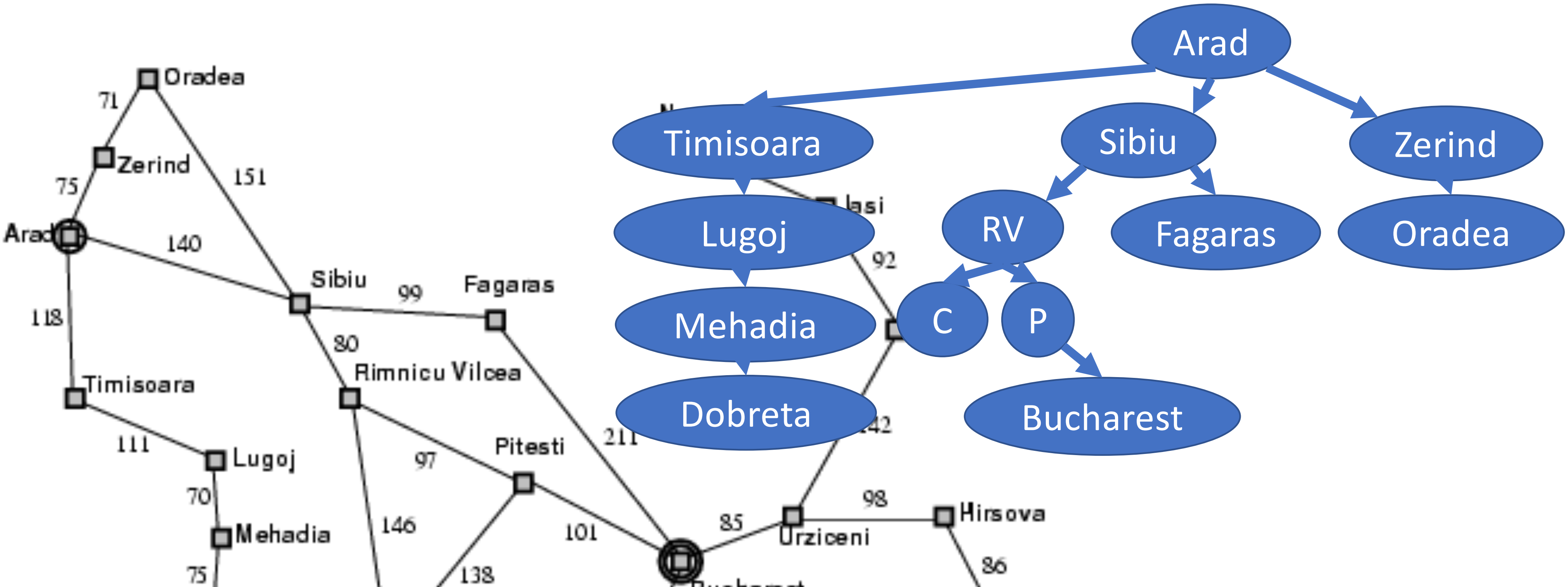
Pitesti:317, Craiova:366, **Dobreta:384**, Bucharest:450

Example of UCS: Romania



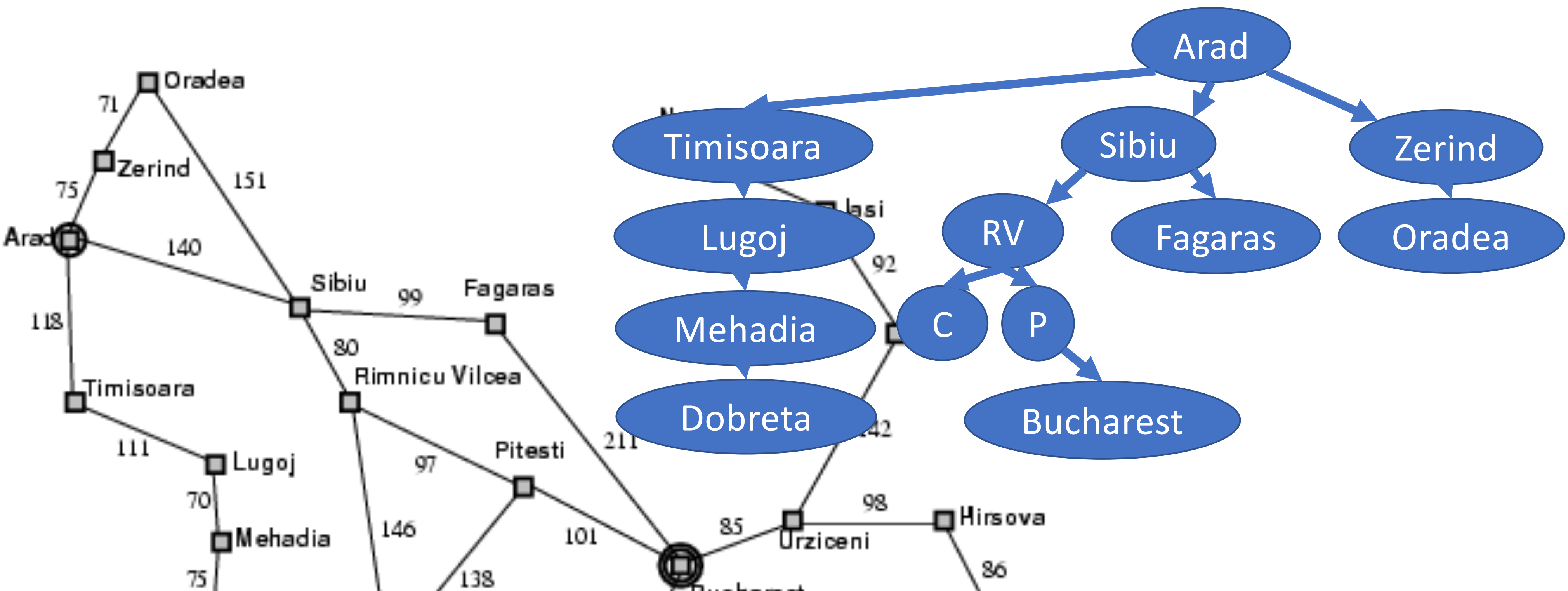
Craiova:366, Dobreta:384, **Bucharest:418**

Example of UCS: Romania



Dobreta:384, Bucharest:418

Example of UCS: Romania



Bucharest:418

GOAL!!!!   GOL!!!!!

Image by Rick Dikeman, GFDL 1996, https://commons.wikimedia.org/wiki/File:Football_iu_1996.jpg

Outline of today's lecture

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Heuristics main idea

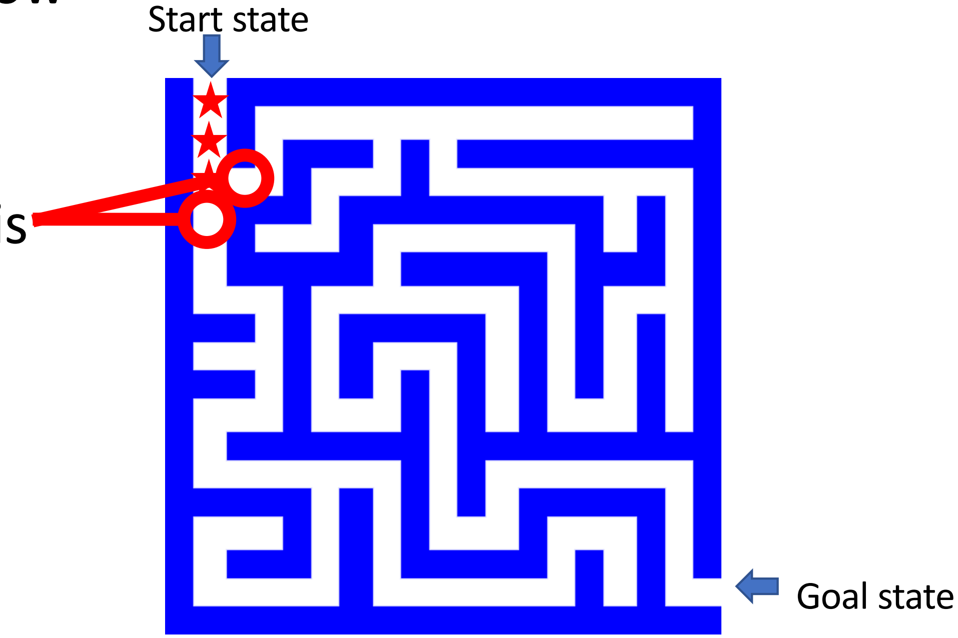
Instead of choosing the node with the smallest total cost so far (UCS),

why not choose the node that's CLOSEST TO GOAL,

and expand that one first?

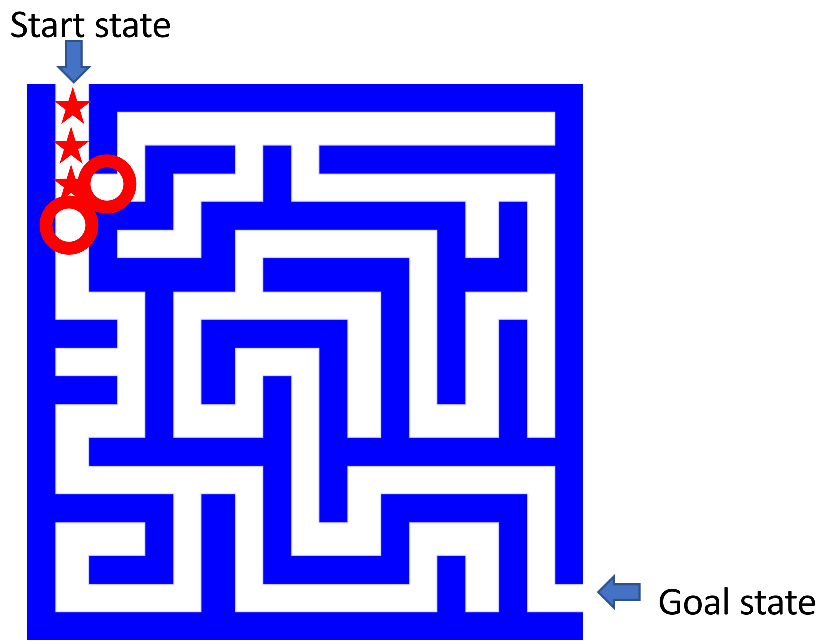
Why not choose the node CLOSEST TO GOAL?

- Answer: because we don't know which node that is!!
- Example: which of these two is closest to goal?



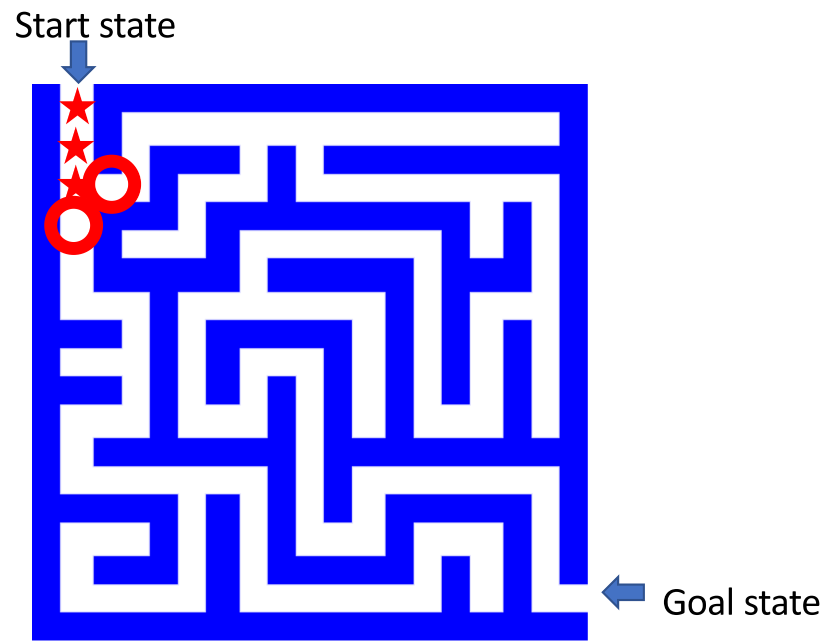
We don’t know which state is closest to goal

- Finding the shortest path is the whole point of the search
- If we already knew which state was closest to goal, there would be no reason to do the search
- Figuring out which one is closest, in general, is a complexity $O\{b^d\}$ problem.



Search heuristics: estimates of distance-to-goal

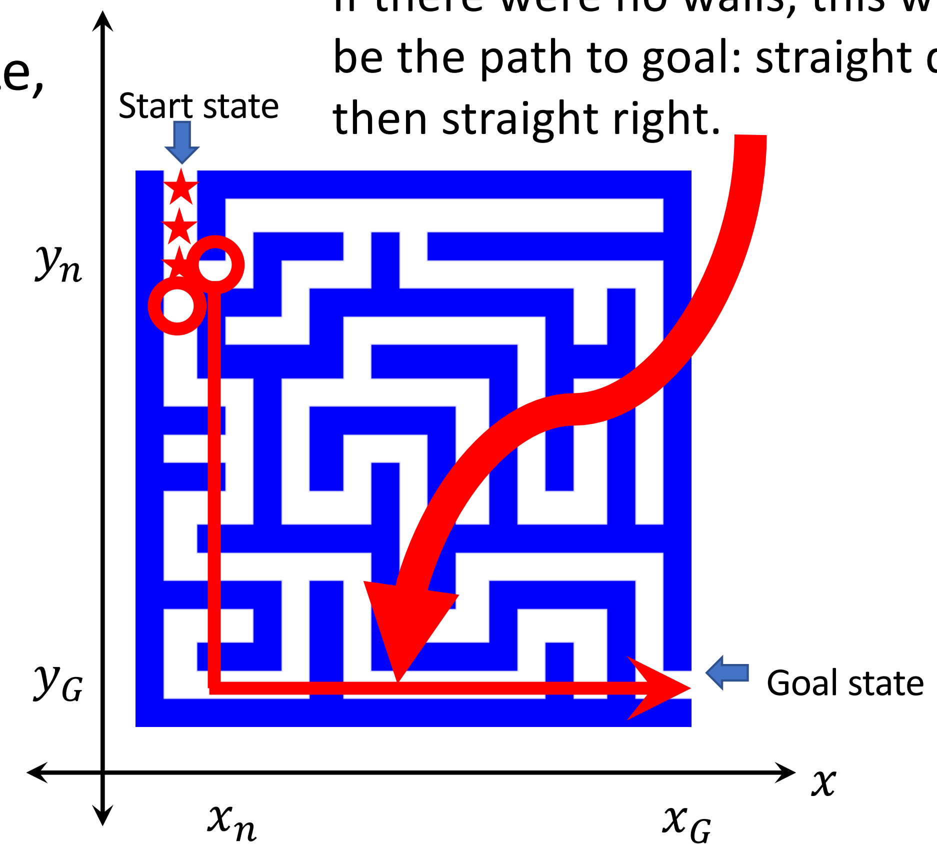
- Often, even if we don't know the distance to the goal, we can estimate it.
- This estimate is called a heuristic.
- A heuristic is useful if:
  1. **Accurate**: $h(n) \approx d(n)$, where $h(n)$ is the heuristic estimate, and $d(n)$ is the true distance to the goal
  2. **Cheap:** It can be computed in complexity less than $O\{b^d\}$



Example heuristic: Manhattan distance

If there were no walls in the maze, then the number of steps from position (x_n, y_n) to the goal position (x_G, y_G) would be

$$h(n) = |x_n - x_G| + |y_n - y_G|$$



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Greedy Best-First Search

Instead of choosing the node with the smallest total cost so far (UCS),

why not choose the node whose

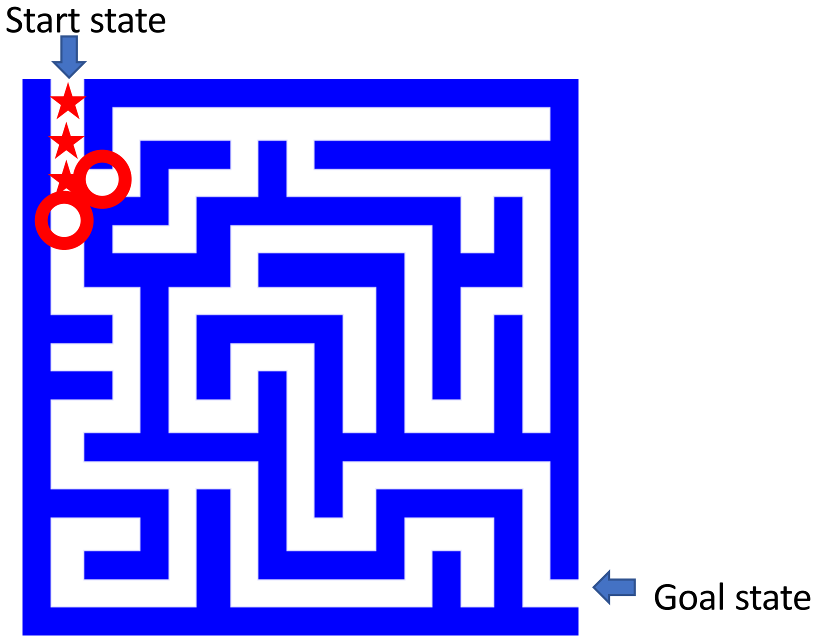
HEURISTIC ESTIMATE

indicates that it might be

CLOSEST TO GOAL?

Greedy Search Example

According to the Manhattan distance heuristic, these two nodes are equally far from the goal, so we have to choose one at random.

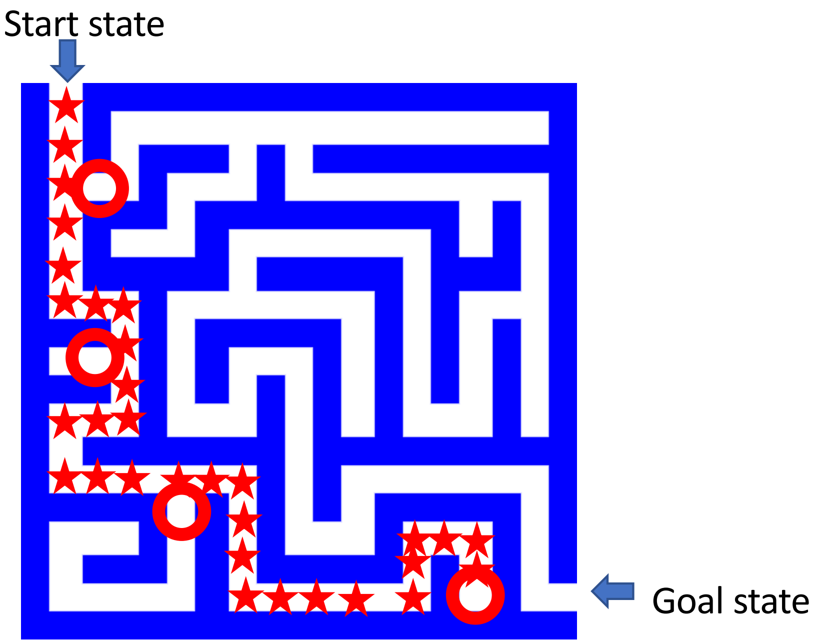


Greedy Search Example

If our random choice goes badly, we might end up very far from the goal.

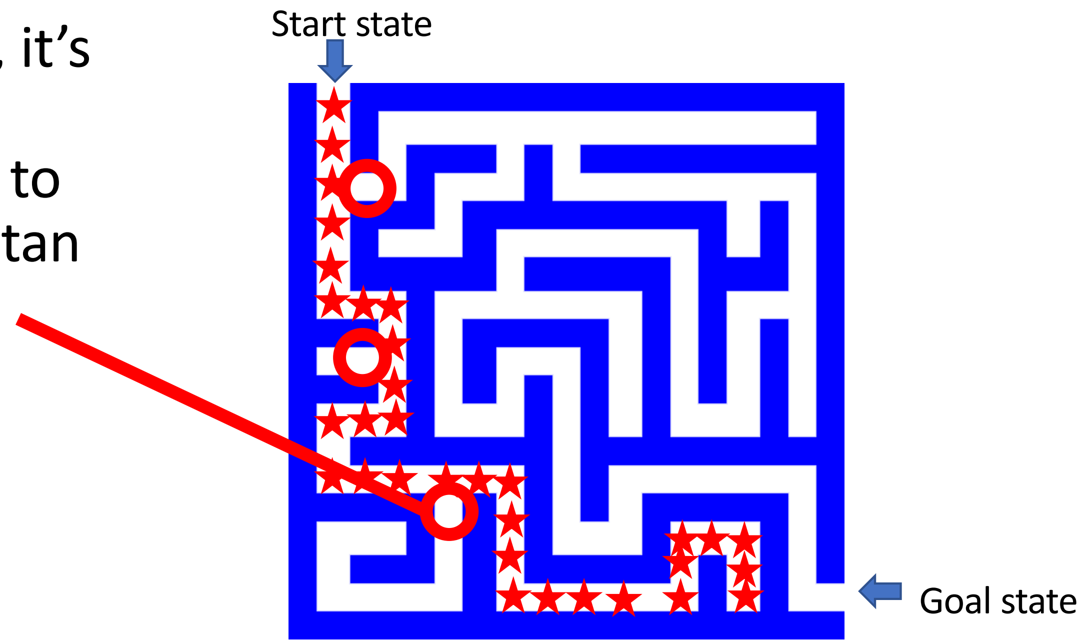
★ = states in the explored set

○ = states on the frontier



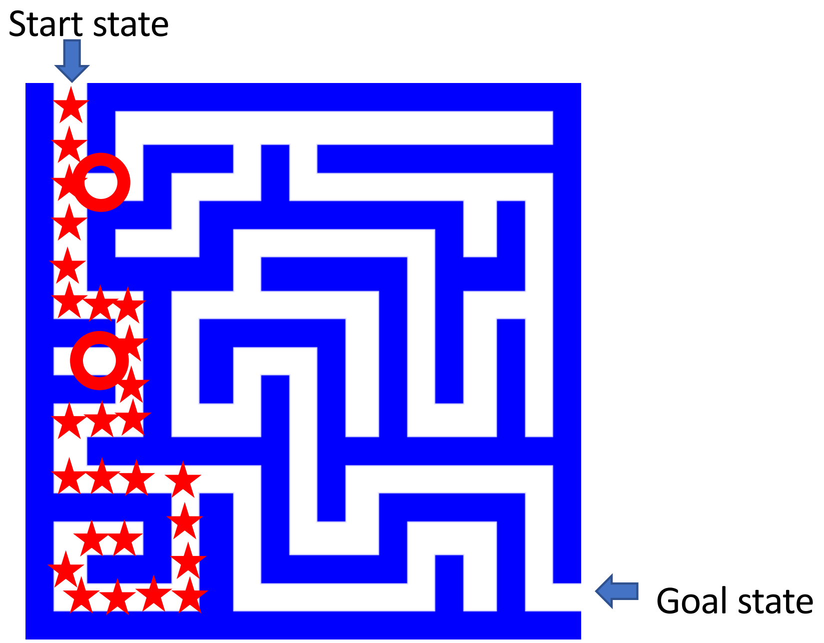
The problem with Greedy Search

Having gone down a bad path, it's very hard to recover, because now, the frontier node closest to goal (according to the Manhattan distance heuristic) is this one:



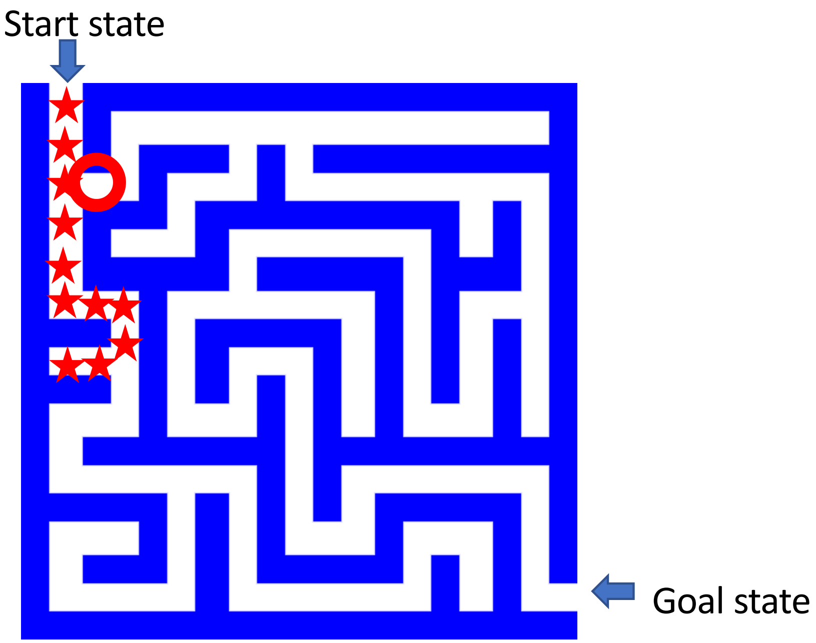
The problem with Greedy Search

That's not a useful path…



The problem with Greedy Search

Neither is that one…



What went wrong?

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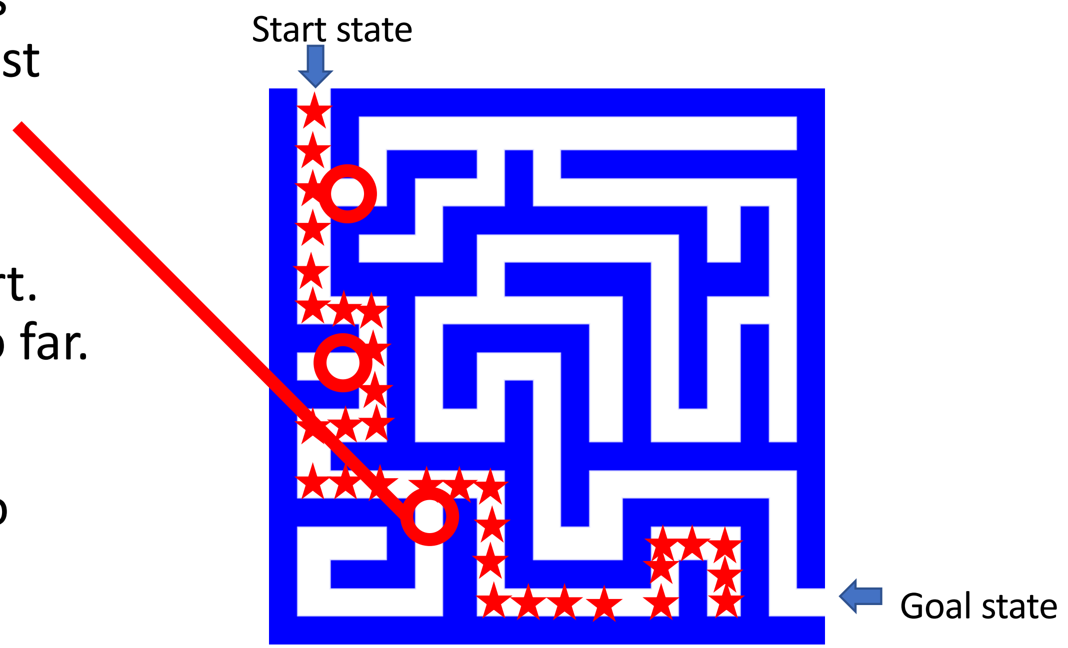
The problem with Greedy Search

Among nodes on the frontier, this one seems closest to goal (smallest $h(n)$, where $h(n) \leq d(n)$).

But it's also farthest from the start. Let's say $g(n)$ = total path cost so far.

So the total distance from start to goal, going through node n, is

$$c(n) = g(n) + d(n) \geq g(n) + h(n)$$



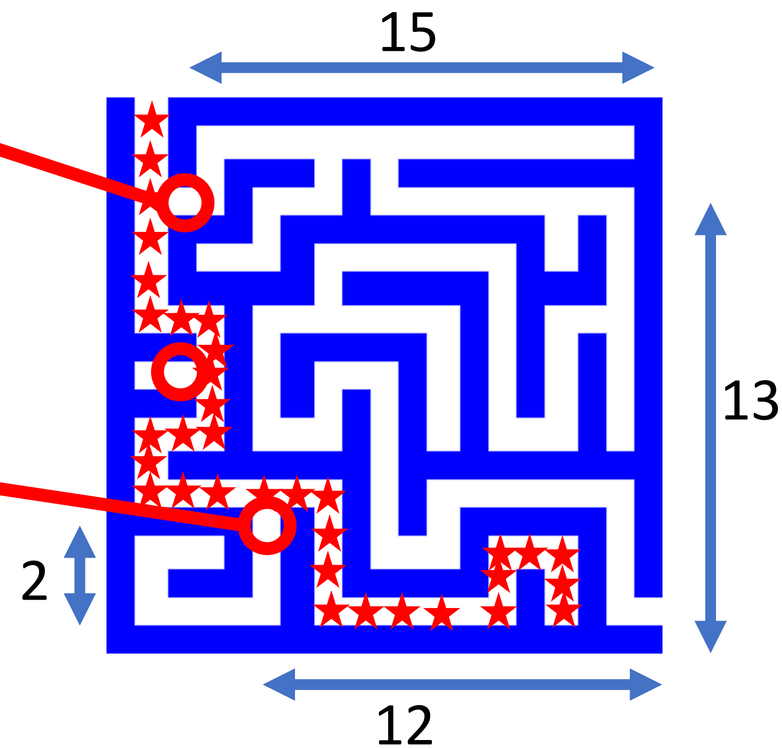
The problem with Greedy Search

Of these three nodes, this one has the smallest $g(n) + h(n)$

$$(g(n) + h(n) = 4\ +\ 28 = 32)$$

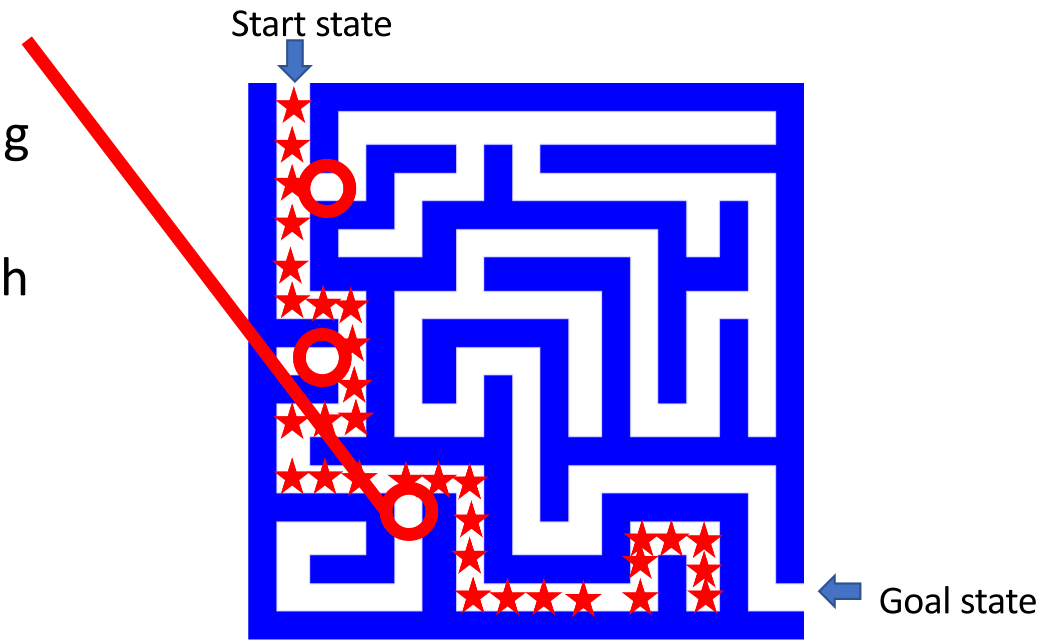
So if we want to find the lowest-cost path, then it would be better to try that node, instead of this one, which has

$$g(n) + h(n) = 21 + 14 = 35$$



A* notation

- $c(n) =$ **<u>cost</u>** of the total path (START,…,n,…,GOAL).
- $d(n) =$ **<u>distance</u>** of the remaining partial path (n,…,GOAL).
- $g(n) =$ **<u>gone-already</u>** on the path so far, (START,…,n).
- $h(n) =$ **<u>heuristic,</u>** $h(n) \leq d(n)$.



$$c(n) = g(n) + d(n) \geq g(n) + h(n)$$

A* Search

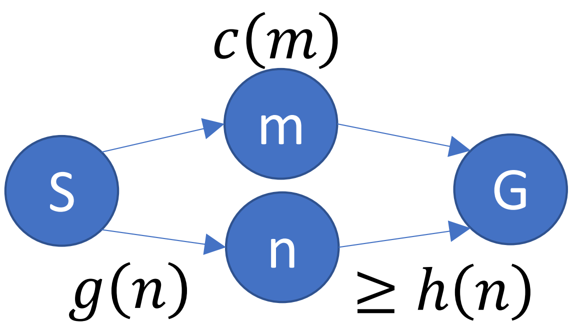
- Idea: avoid expanding paths that are already expensive
- The **evaluation function** $f(n)$ is the estimated total cost of the path through node n to the goal:

$$f(n) = g(n) + h(n)$$

$g(n)$: cost so far to reach n (path cost)
$h(n)$: estimated cost from n to goal (heuristic)

- This is called A* search if and only if the heuristic, h(n), is **admissible**. The word "admissible" just means that $h(n) \leq d(n)$, and therefore, $f(n) \leq c(n)$.

Admissible heuristic



- Suppose we've found one path to G; the path goes through node m. Since we've calculated the whole path, we know its total path cost to be $c(m)$.
- For every other node, n, we don't know $c(n)$, but we know $f(n) = g(n) + h(n)$, and we know that

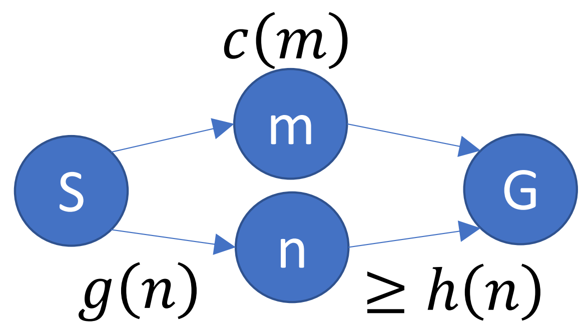
$$c(n) \geq f(n)$$

- Therefore we know that

$$\begin{aligned} IF \quad & f(n) \geq c(m) \\ THEN \; & c(n) \geq c(m) \end{aligned}$$

- So if $f(n) \geq c(m)$ for every node n that's still in the frontier, then we know that m is the best path.

A* Search

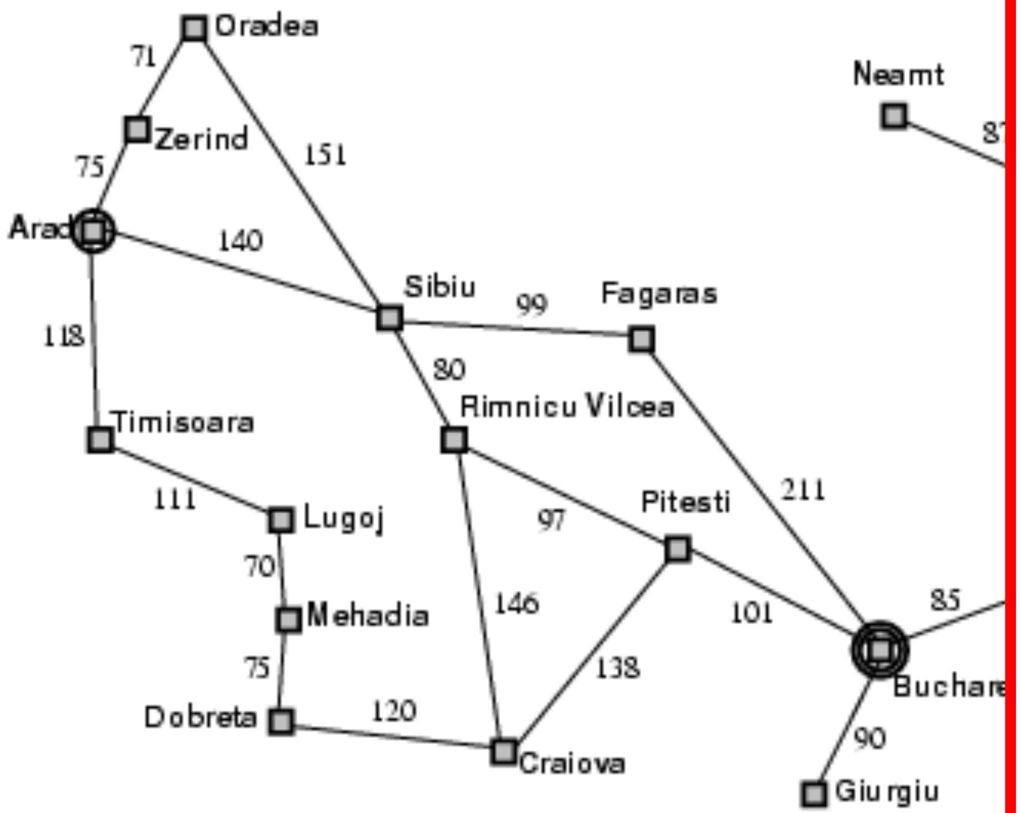


Definition: A* SEARCH

- If $h(n)$ is **admissible** $(d(n) \geq h(n))$**,** and
- if the frontier is a priority queue sorted according to $g(n) + h(n)$, then
- the FIRST path to goal uncovered by the tree search, path m, is guaranteed to be the SHORTEST path to goal

  ($h(n) + g(n) \geq c(m)$ for every node n that is not on path m)

Example of A*: Romania

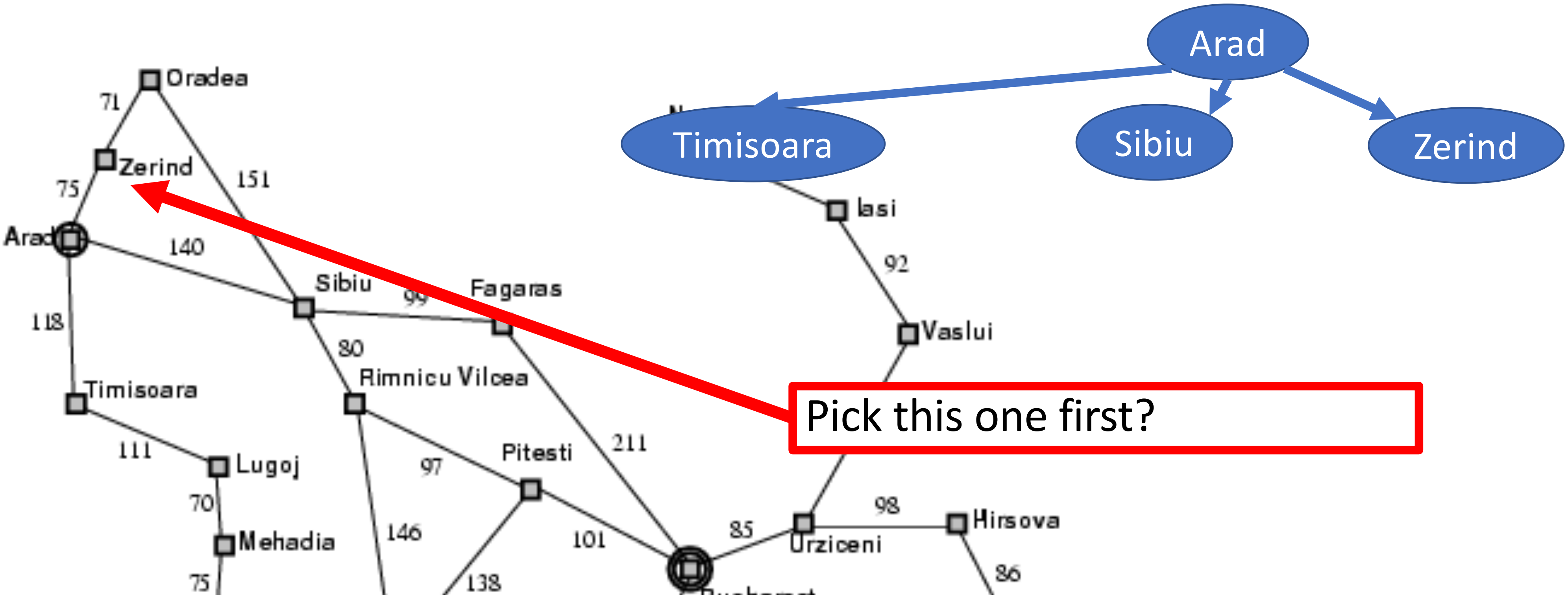


Suppose we don't know the distance from Sibiu to Bucharest on highways, but we DO know the distance "as the crow flies."

$h(n)$ = Euclidean distance (as the crow flies)

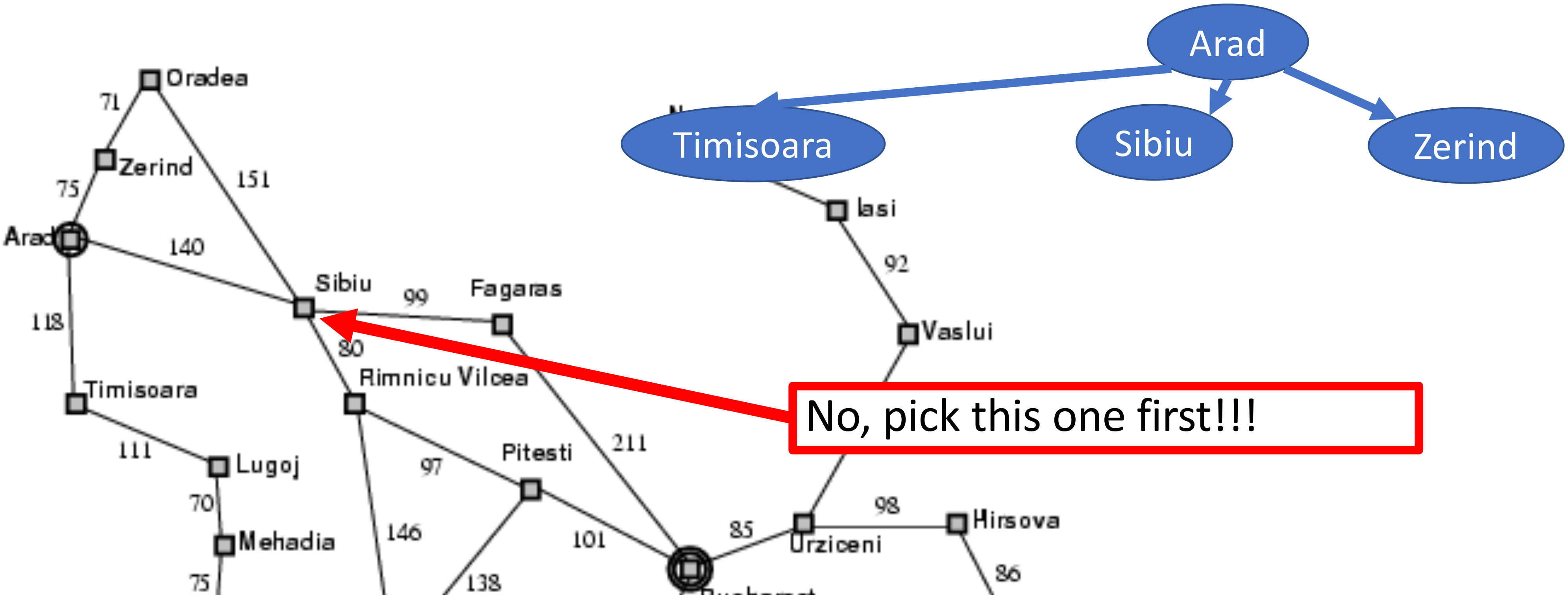
- Sibiu: h(n) = 260km
- Timisoara: h(n) = 410km
- Zerind: h(n) = 422km

Romania using UCS



Zerind:75, Timisoara:118, Sibiu:140

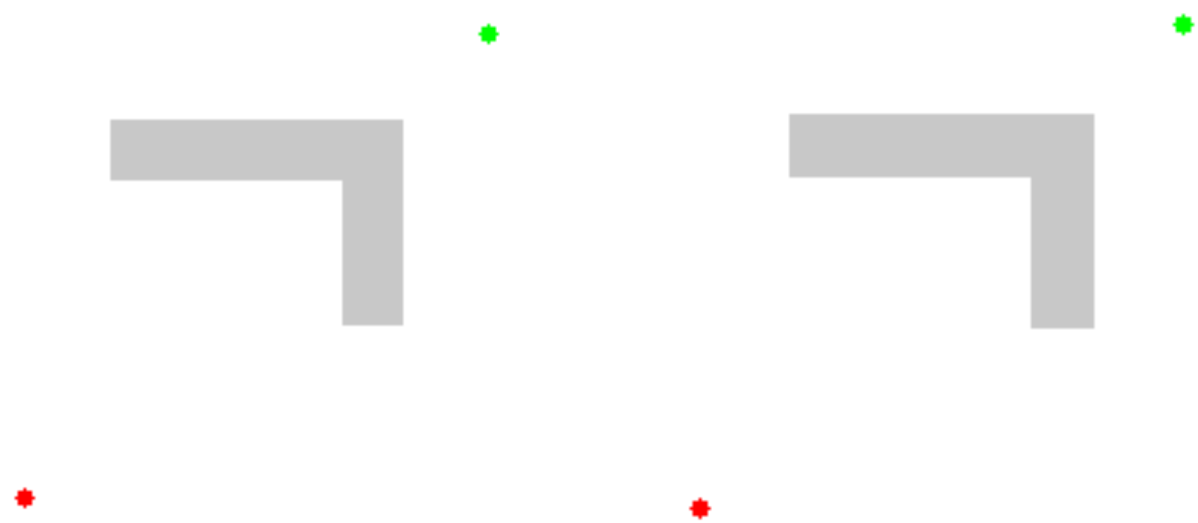
Romania using A*



Sibiu:140+260=400, Zerind:75+422=495, Timisoara:118+410=528

BFS vs. A* Search Example

The heuristic h(n)=Euclidean distance favors nodes on the main diagonal. Those nodes all have the same g(n)+h(n), so A* evaluates them first.



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