Slide 5 : Σ should be Π

Slide 9:

Numerator ⟷ Denominator

Slide 27: ∂

missing last line

Slide $D = BA$

Slide 40 = =

$$y = \frac{dx}{dt}$$

LINEAR

$$x_1 \longrightarrow y_1 = \frac{dx_1}{dt}$$

$$x_2 \longrightarrow y_2 = \frac{dx_2}{dt}$$

$$x = x_1 + x_2 \longrightarrow y = \frac{dx}{dt}$$

$$= \frac{d}{dt}\left(x_1(t) + x_2(t)\right)$$

$$= \frac{dx_1}{dt} + \frac{dx_2}{dt}$$

$$= y_1(t) + y_2(t)$$ ✓

LINEAR

SHIFT-INVARIANT

$$x_1(t) \longrightarrow y_1(t) = \frac{dx_1}{dt}$$

$$x(t) = x_1(t - t_0) \qquad x_1'(t) \equiv \frac{dx_1}{dt}$$

$$\longrightarrow y(t) = \frac{dx}{dt}$$

$$= \frac{d}{dt}\, x_1(t - t_0)$$

$$= \frac{d x_1(t-t_0)}{d(t-t_0)} \cdot \frac{d(t-t_0)}{dt}$$

$$= x_1'(t-t_0) \cdot 1$$

$$= x_1'(t-t_0) = y_1(t-t_0)$$

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System

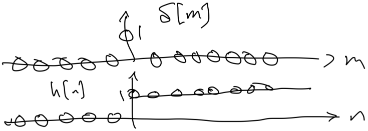
$$x[n] \rightarrow \boxed{\mathcal{H}} \rightarrow y[n]$$

$$\boxed{y[n] = \sum_{m=-\infty}^{n} x[m]} \overset{②?}{=} x[n] * h[n]$$

① What is $h[n]$?

$$\delta[n] \rightarrow \boxed{\mathcal{H}} \rightarrow h[n]$$

$$h[n] = \sum_{m=-\infty}^{n} \delta[m]$$



$$h[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$= u[n]$ unit step function or "Heaviside"

② Is $y[n] = h[n] * x[n]$?

$$h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$= \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

$$\sum_{m=0}^{\infty} x[n-m] = \sum_{m'=-\infty}^{n} x[m']$$

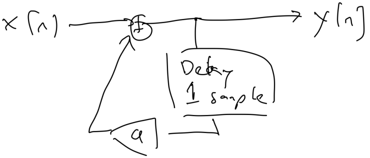
$m' = n - m$

$$\sum_{m:\ n-m \geq 0} x[m] = \sum_{m=-\infty}^{n} x[m]$$

$m - n \leq 0$

$m \leq n$

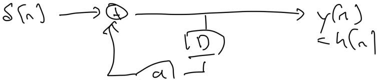
EXAMPLE #2



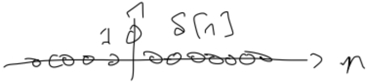
$$y[n] = x[n] + a\,y[n-1]$$

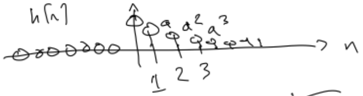
① What is $h[n]$?

② Is $y[n] = x[n] * h[n]$?



$$h[n] = \delta[n] + a\,h[n-1]$$





$$h[n] = \begin{cases} 0 & n < 0 \\ a^n & n \geq 0 \end{cases}$$

$$h[n] = a^n u[n]$$

② $y[n] = x[n] * a^n u[n]$?

$$\overset{?}{=} \sum^{\infty} h[m]\, x[n-m]$$

$m=-\infty$

$$= \sum_{m=0}^{\infty} a^{m}\, x[n-m]$$

$$\overset{?}{=} x[n] + a\,y[n-1] \quad ?$$

$$x[n] + \sum_{m=1}^{\infty} a^{m}\, x[n-m]$$

$$= x[n] + a \sum_{m=1}^{\infty} a^{m-1}\, x[n-m]$$

$$m' = m-1$$

$$m'+1 = m$$

$$y[n] = x[n] + a \sum_{m'=0}^{\infty} a^{m'}\, x[n-m'-1]$$

$$= x[n] + a\,y[n-1]$$

Other examples

① Reverberation

$$y[n] = x[n] + \sum_{k} a_k\, y[n-n_k]$$

n_k = delay of kth echo

a_k = amplitude of kth echo

② Flutes & Speech & other resonators?

Same as above,

a_k complex

$a_k = a^{*}_{-k}$

③ Local averaging

$$y[n] = \sum_{m} h[m]\, x[n-m]$$

imp-lse resp = h[n]

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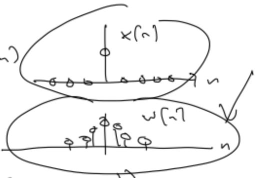
$x[n] = \delta[n]$



$w[n] = e^{-\frac{n^2}{2}}$

$g(x) = \sigma(\max(w[n] * x[n]))$

$y = 1$

$\mathcal{L} = -y \log(x) - (1-y)\ln(1-g(x))$

$= -\log(x)$

Find $\boxed{\frac{\partial \mathcal{L}}{\partial w[n]}}$

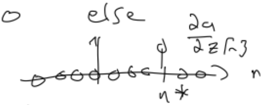
$z[n] = w[m] * x[n]$

$= \sum_{m} w[m]\, x[n-m]$

$\frac{\partial z[n]}{\partial w[m]} = x[n-m]$

$a = \max_n(z[n])$

$\frac{\partial a}{\partial z[n]} = \begin{cases} 1 & n = \text{argmax } z[n] \\ 0 & \text{else} \end{cases}$



$g = \sigma(a)$

$\frac{\partial g}{\partial a} = \sigma(a)(1-\sigma(a)) = g(1-g)$

$\mathcal{L} = -\ln g$

$\frac{\partial \mathcal{L}}{\partial g} = -\frac{1}{g}$

$\frac{\partial \mathcal{L}}{\partial w[m]} = \sum_n \frac{\partial \mathcal{L}}{\partial g} \frac{\partial g}{\partial a} \frac{\partial a}{\partial z[n]} \frac{\partial z[n]}{\partial w[m]}$

$= \sum_n -\frac{1}{\cancel{g}} \cancel{g}(1-g)\, \underline{\delta[n-n^*]}\, x[n-m]$

$= -(1-g)\, x[n^* - m]$

$n^* = \arg\max\, z[n] = 0$

$z[n] = w[n] * x[n] = w[n]$

$$\frac{\partial \mathcal{L}}{\partial w[m]} = \underline{\underline{(g-1)}}\, x[-m]$$

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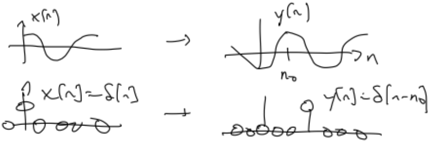
$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = x[n-n_0]$$



$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$= \sum_{m=-\infty}^{\infty} \delta[m-n_0]\, x[n-m]$$

$$= x[n-n_0]$$

$x[n] = e^{j\omega n} \longrightarrow y[n] = ?$

① FORMULA

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega m}$$

$$= \sum_{m=-\infty}^{\infty} \delta[m-n_0]\, e^{-j\omega m}$$

$$= e^{-j\omega n_0}$$

② DIRECT

$$x[n] = e^{j\omega n} \longrightarrow \boxed{\mathcal{H}} \rightarrow y[n] = x[n-n_0]$$

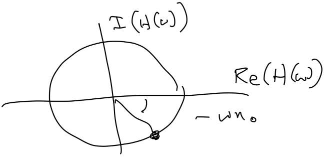
$$= e^{j\omega(n-n_0)}$$

$$= e^{-j\omega n_0}\, e^{j\omega n}$$

Magnitude

$$|H(\omega)| = |e^{-j\omega n_0}|$$

$$= |\cos(-\omega n_0) + j\sin(-\omega n_0)|$$



$$|H(\omega)| = \sqrt{\cos^2(-\omega n_0) + \sin^2(-\omega n_0)}$$

$$= \sqrt{\cos^2\theta + \sin^2\theta} \qquad \theta = -\omega n_0$$

$$= \underline{1} \leftarrow$$

$$\measuredangle H(\omega) = -\omega n_0 \leftarrow$$

$$x[n] = \cos(\omega n) \rightarrow \boxed{?H} \rightarrow y[n]$$

① FORMULA

$$y[n] = |H(\omega)| \cdot \cos(\omega n + \measuredangle H(\omega))$$

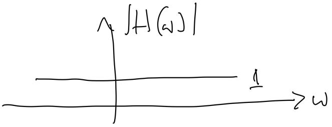
$$= \underline{1} \cdot \cos(\omega n - \omega n_0)$$

$$= \cos(\omega(n - n_0))$$

② DIRECT

$$y[n] = x[n - n_0]$$

$$= \cos(\omega(n - n_0))$$



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$a_k b_k$ $\quad a = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$

$$P = b^T a - a^T C a$$

Maximize P

$$\frac{\partial P}{\partial a} = 0$$

$$\frac{\partial P}{\partial a} = \underline{b^T - a^T}(C + C^T)$$

$$= 0$$

$$b^T = a^T(C + C^T)$$

$$b^T(C+C^T)^{-1} = a^T \underline{(C+C^T)}\,\underline{(C+C^T)^{-1}}$$

$$b^T(C+C^T)^{-1} = a^T$$

$$(C+C^T)^{-1} b = a$$

$$\frac{\partial}{\partial a}(a^T C a)$$

$$= \frac{\partial}{\partial a} \underline{sg(a^T C)\, a}$$

$$+ \left(\frac{\partial}{\partial a^T} a^T sg(Ca)\right)^T$$

$$= a^T C + (Ca)^T$$

$$= a^T C + a^T C^T$$

$$\frac{\partial P}{\partial a} = b^T - a^T C - a^T C^T$$

$$a = (C + C^T)^{-1} b$$

Problem 21

$$x[n] = \begin{cases} \cos(\omega_0 n) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X[k] = \text{DFT}(x[n])$$

$$= \text{DTFT}(x[n]w[n])\Big|_{\omega = \frac{2\pi k}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}\Big|_{\omega = \frac{2\pi k}{N}}$$

$$= \sum_{n=-\infty}^{\infty} y[n]\, w_R[n]\, e^{-j\omega n}$$

if $w_R[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{else} \end{cases}$

$y[n] = \cos(\omega_0 n)$ for all time

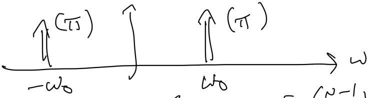
$x[n] = y[n]\, w_R[n]$

$$X(\omega) = \frac{1}{2\pi} Y(\omega) * W_R(\omega)$$

$$y[n] = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$Y(\omega) = \frac{1}{2}\left(2\pi\,\delta(\omega-\omega_0) + 2\pi\,\delta(\omega+\omega_0)\right)$$

$$= \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

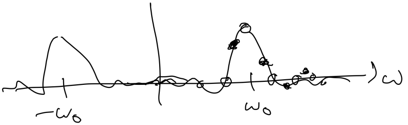


$$W_R(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}\, e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$X(\omega) = \frac{1}{2\pi} Y(\omega) * W_R(\omega)$$

$$= \frac{1}{2\pi}\left(\pi W_R(\omega-\omega_0) + \pi W_R(\omega+\omega_0)\right)$$

$$= \frac{1}{2} W_R(\omega-\omega_0) + \frac{1}{2} W_R(\omega+\omega_0)$$

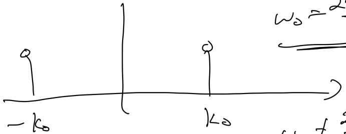


$$X[k] = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}}$$

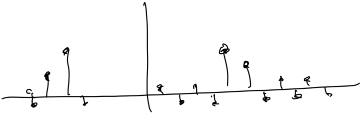
$$= \frac{1}{2} W_R\left(\frac{2\pi k}{N} - \omega_0\right)$$

$$+ \frac{1}{2} W_R\left(\frac{2\pi k}{N} + \omega_0\right)$$

$$\underline{\omega_0 = \frac{2\pi k_0}{N}}$$



$$\omega_0 \neq \frac{2\pi k_0}{N}$$



Problem 23

$$X_m[k] \longrightarrow x[n]$$

Initial estimate

$$X_m[k] = \begin{array}{cc|cc} & & \multicolumn{2}{c}{m} \\ & & \text{even} & \text{odd} \\ \hline k & 0 & 2 & 1 \\ & 1 & 1 & 2 \end{array}$$

$$x_0[k] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad X_1[k] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

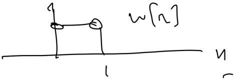
$X_2[k] = \begin{bmatrix}2\\1\end{bmatrix}$ $\ldots$

(a) $x[n] = \text{ISTFT}(X_m[k])$

FIND $x[n]$ FOR $n \geq 1$

$\sum_m w[n-m] = 2$

$$w[n] = \begin{cases}1 & 0 \leq n \leq 1\\ 0 & \text{else}\end{cases}$$



$$x[n] = \frac{\sum_m \frac{1}{N}\sum_{k=0}^{N-1} X_m[k]\, e^{j\omega_k(n-m)}}{\sum_m w[n-m]}$$

$$= \frac{\sum_m \text{IDFT}(X_m[k])\Big|_{n-m}}{\sum_m w[n-m]}$$

$$= \frac{1}{2}\sum_m \text{IDFT}(X_m[k])\Big|_{n-m}$$

$$\text{IDFT}\begin{pmatrix}X_m[0]=2\\ X_m[1]=1\end{pmatrix}$$

$$= \frac{1}{N}\sum_k X_m[k]\, e^{j\frac{2\pi k n}{N}}$$

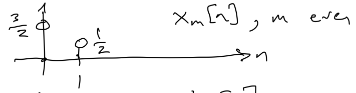
$$= \frac{1}{2}\sum_k X_m[k]\, e^{j\pi k n}$$

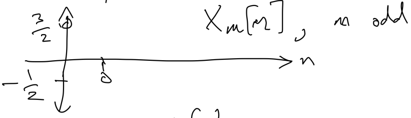
$$= \frac{1}{2}\left(X_m[0]e^{0} + X_m[1]e^{j\pi n}\right)$$

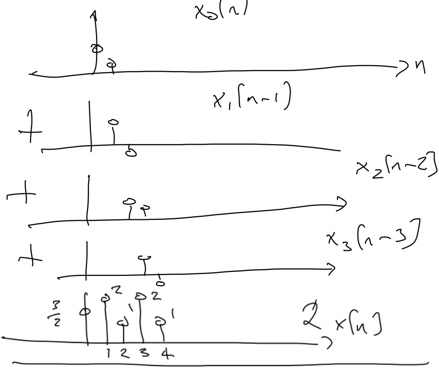
$$= \frac{1}{2}\left(X_m[0] + X_m[1](-1)^n\right)$$

$= \frac{1}{2}\left(2 + (-1)^n\right)$ m even

$= \frac{1}{2}\left(1 + 2(-1)^n\right)$ m odd







$$x[n] = \begin{cases} 3/\cancel{4} & n = 0 \\ 2/2 & n \text{ even} \\ 1/2 & n \text{ odd} \end{cases}$$

FOR $n \geq 1$

$$\boxed{x(n) = \begin{cases} 1 & n \text{ even} \\ \frac{1}{2} & n \text{ odd} \end{cases}}$$

(b) $\tilde{X}_m[k] = \text{STFT}(x[n])$

FIND FOR $m \geq 1$

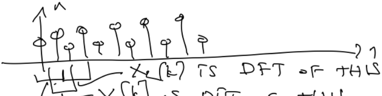
$$\tilde{X}_m[k] = \sum_n w[n-m]\, x[n]\, e^{-j\omega(n-m)}$$

$$\omega = \frac{2\pi k}{N}$$

$$= \sum_{n'} w[n']\, x[n'+m]\, e^{-j\omega n'}$$

$n' = n - m$

$$= \mathrm{DFT}\left(w[n]\, x[n+m]\right)$$



$X_2[k]$ IS DFT OF THIS

$X_1[k]$ IS DFT OF THIS

$X_0[k]$ IS DFT OF THIS

$$X_1[k] = \mathrm{DFT}\left(1 \quad \tfrac{1}{2} \qquad x_1[n] \right)$$

$$X_2[k] = \mathrm{DFT}\left(\tfrac{1}{2} \quad 1 \qquad x_2[n] \right)$$

$$X_3[k] = \mathrm{DFT}\left(1 \quad \tfrac{1}{2} \qquad x_3[n] \right)$$

$$X_1[k] = \sum_{n=0}^{1} x_1[n]\, e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{1} x_1[n]\, e^{-j\pi k n}$$

$$= x_1[0] + x_1[1](-1)^k$$

$$= 1 + \frac{1}{2}(-1)^k$$

$$= \left(\begin{cases} \frac{3}{2} & k=0 \\ \frac{1}{2} & k=1 \end{cases} \right)$$

$$X_2[k] = x_2[0] + x_2[1](-1)^k$$

$$= \frac{1}{2} + (-1)^k$$

$$= \left(\begin{cases} \frac{3}{2} & k=0 \\ -\frac{1}{2} & k=1 \end{cases} \right)$$

[c]

Keep those phases,

but use original magnitude

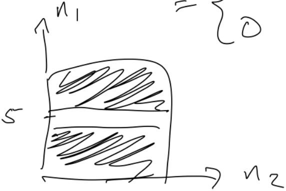
$$X_1[k] = \begin{cases} 1 & k=0 \\ 2 & k=1 \end{cases}$$

$$X_2[k] = \begin{cases} 2 & k=0 \\ -1 & k=1 \end{cases}$$

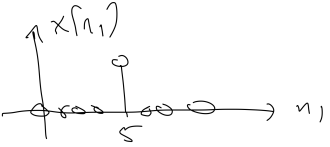
Problem 7

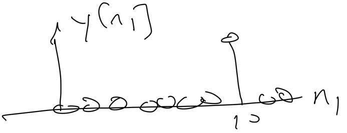
$$x[n_1, n_2] = \delta[n_1 - 5]$$

$$= \begin{cases} 1 & n_1 = 5 \\ 0 & \text{otherwis} \end{cases}$$



$$y[n_1, n_2) = \begin{cases} x\left[\frac{n_1}{2}, \frac{n_2}{2}\right] & \frac{n_1}{2}, \frac{n_2}{2} \text{ integers} \\ 0 & \text{else} \end{cases}$$





$$z[n_1,n_2] = y[n_1,n_2] * h[n_1,n_2]$$

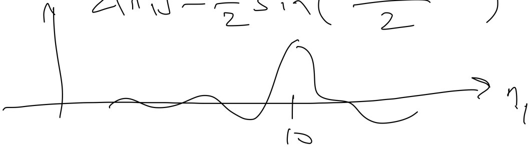
$$H(\omega_1,\omega_2) = \begin{cases} 1 & |\omega_1| < \frac{\pi}{2},\ |\omega_2| < \frac{\pi}{2} \\ 0 & \text{else} \end{cases}$$

$$= H(\omega_1)\, H(\omega_2)$$

$$H(\omega_1) = \begin{cases} 1 & |\omega_1| < \frac{\pi}{2} \\ 0 & \text{else} \end{cases}$$

$$h[n_1,n_2] = \left(\frac{1}{2}\,\mathrm{sinc}\left(\frac{\pi n_1}{2}\right)\right) \cdot \boxed{\left(\frac{1}{2}\,\mathrm{sinc}\left(\frac{\pi n_2}{2}\right)\right)}$$

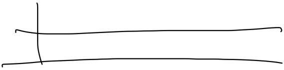
$$z[n_1] = \frac{1}{2}\sin\left(\frac{\pi(n_1-10)}{2}\right)$$

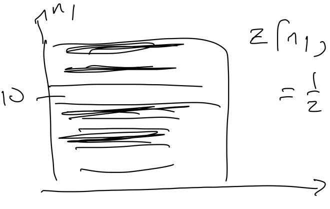


$$y[n_1,n_2] = 1 \qquad n_1 = 10$$

$$y[n_1,n_2] = 0 \qquad n_1 \neq 10$$

$$z[n_1,n_2] = 1 \qquad n_1 = 10$$





$$z[n_1,n_2] = \frac{1}{2}\,\mathrm{sinc}\left(\frac{\pi n_1}{2}\right)$$