ECE 417 Multimedia Signal Processing
Solutions to Homework 2

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Assigned: Tuesday, 9/7/2021; Due: Thursday, 9/16/2021
Reading: **Strang, Section 6.1**

Problem 2.1

Let A be a 2×2 matrix, and let x be one of its elements. All of its other elements are known, and are given as:

$$A = \begin{bmatrix} x & 3 \\ -1 & 2 \end{bmatrix} \tag{2.1-1}$$

The eigenvalues of A are given by

$$\lambda = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{2.1-2}$$

for some particular values of a, b, and c. Find a, b, and c, in terms of x, such that Equation (2.1-2) gives the eigenvalues of A.

Solution:

$$a = 1$$
$$b = -(x+2)$$
$$c = 2x + 3$$

Problem 2.2

Let A be a 2×2 matrix, and let x be one of its elements. All of its other elements are known, and are given as:

$$A = \begin{bmatrix} x & 3 \\ -1 & 2 \end{bmatrix} \tag{2.2-1}$$

Suppose that you are given one of its eigenvalues, λ, and you want to find the corresponding eigenvector. As you know, the scale of an eigenvector is arbitrary, so let's arbitrarily set its first element to 1: $\vec{v} = [1, v_2]^T$. Solve for its second element, v_2, in terms of λ.

Solution: Setting $A\vec{v} = \lambda\vec{v}$ gives two equations in one unknown: $x + 3v_2 = \lambda$, and $-1 + 2v_2 = \lambda v$. These will give the same answer if λ is an eigenvalue:

$$v_2 = \frac{\lambda - x}{3} = \frac{1}{2 - \lambda}$$

Problem 2.3

Suppose that A is a tall thin matrix (more rows than columns). Suppose that $A^\dagger = (A^TA)^{-1}A^T$ is its pseudo-inverse, and that $\vec{v}^* = A^\dagger\vec{b}$. Show that $\vec{v}^*$ is the minimum-squared error solution to the equation $A\vec{v} \approx \vec{b}$, i.e., show that $\vec{v}^*$ minimizes

$$E = \|A\vec{v} - \vec{b}\|_2^2$$

Solution:

$$\begin{aligned} E &= \|A\vec{v} - \vec{b}\|_2^2 \\ &= (A\vec{v} - \vec{b})^T(A\vec{v} - \vec{b}) \\ &= \vec{v}^TA^TA\vec{v} - 2\vec{v}^TA^T\vec{b} + \vec{b}^T\vec{b} \end{aligned}$$

Differentiating w.r.t. $\vec{v}$ gives

$$\nabla_{\vec{v}}E = 2A^TA\vec{v} - 2A^T\vec{b}$$

Setting this to zero gives $\vec{v}^* = (A^TA)^{-1}A^T\vec{b}$.

Problem 2.4

Suppose that A is a short fat matrix (more columns than rows). Suppose that $A^\dagger = A^T(AA^T)^{-1}$ is its pseudo-inverse, and that $\vec{v}^* = A^\dagger\vec{b}$. Show that $\vec{v}^*$ satisfies the equation $A\vec{v}^* = \vec{b}$.

Solution:

$$\begin{aligned} A\vec{v}^* &= AA^T(AA^T)^{-1}\vec{b} \\ &= \vec{b} \end{aligned}$$