Lecture 31: Damped Resonators

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis

1. Review: Lossless resonators
2. Impulse response of a damped resonator
3. Bandwidth
4. Underdamped, Overdamped, and Critically-damped filters
5. Summary

Outline

1. Review: Lossless resonators
2. Impulse response of a damped resonator
3. Bandwidth
4. Underdamped, Overdamped, and Critically-damped filters
5. Summary

Review: Poles and Zeros

A first-order autoregressive filter,

$$y[n] = x[n] - bx[n-1] + ay[n-1],$$

has the impulse response and transfer function

$$h[n] = a^n u[n] - ba^{n-1} u[n-1] \leftrightarrow H(z) = \frac{1 - bz^{-1}}{1 - az^{-1}},$$

where a is called the **pole** of the filter, and b is called its **zero**.

Impulse Response of a Second-Order All-Pole Filter

A general second-order all-pole filter has the system function

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} = \frac{1}{1 - (p_1 + p_1^*)z^{-1} + p_1 p_1^* z^{-2}}$$

Its impulse response is

$$\begin{aligned} h[n] &= g_1 p_1^n u[n] + g_1^* (p_1^*)^n u[n] \\ &= \frac{\Im(p_1^{n+1})}{\Im(p_1)} u[n] \end{aligned}$$

Impulse Response of a Lossless Resonator

A lossless resonator has its poles at $p_1 = e^{j\omega_1}$ and $p_2 = e^{-j\omega_1}$, so

$$H(z) = \frac{1}{1 - 2\cos(\omega)z^{-1} + z^{-2}}$$

Its impulse response is

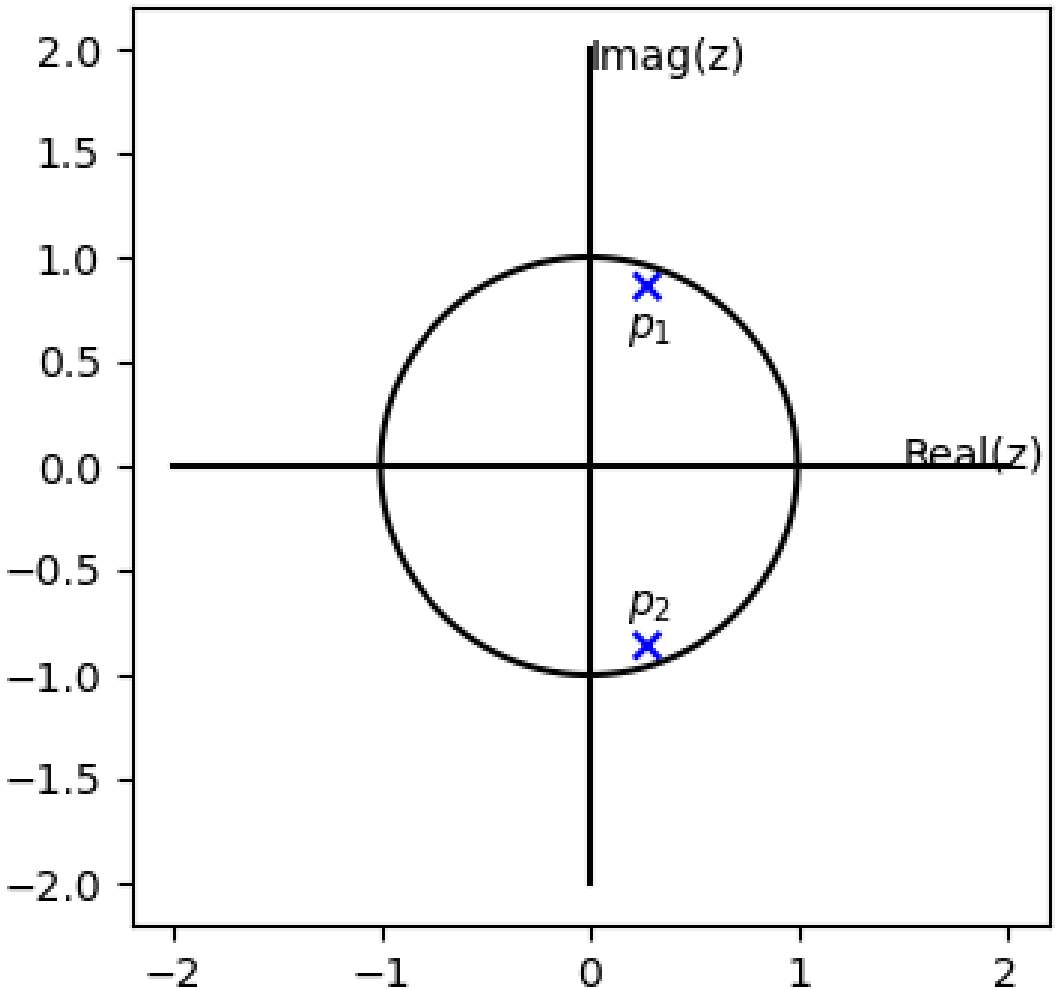
$$\begin{aligned} h[n] &= g_1 p_1^n u[n] + g_1^* (p_1^*)^n u[n] \\ &= \frac{\sin(\omega_1(n+1))}{\sin(\omega_1)} u[n] \end{aligned}$$

Outline

1. Review: Lossless resonators
2. Impulse response of a damped resonator
3. Bandwidth
4. Underdamped, Overdamped, and Critically-damped filters
5. Summary

Poles of a damped resonator

Now, let's suppose we put p_1 inside the unit circle, $p_1 = e^{-\sigma_1 + j\omega_1}$.



Difference equation of a damped resonator

The system function for this filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2e^{-\sigma_1}\cos(\omega_1)z^{-1} + e^{-2\sigma_1}z^{-2}}$$

Solving for $y[n]$, we get the difference equation:

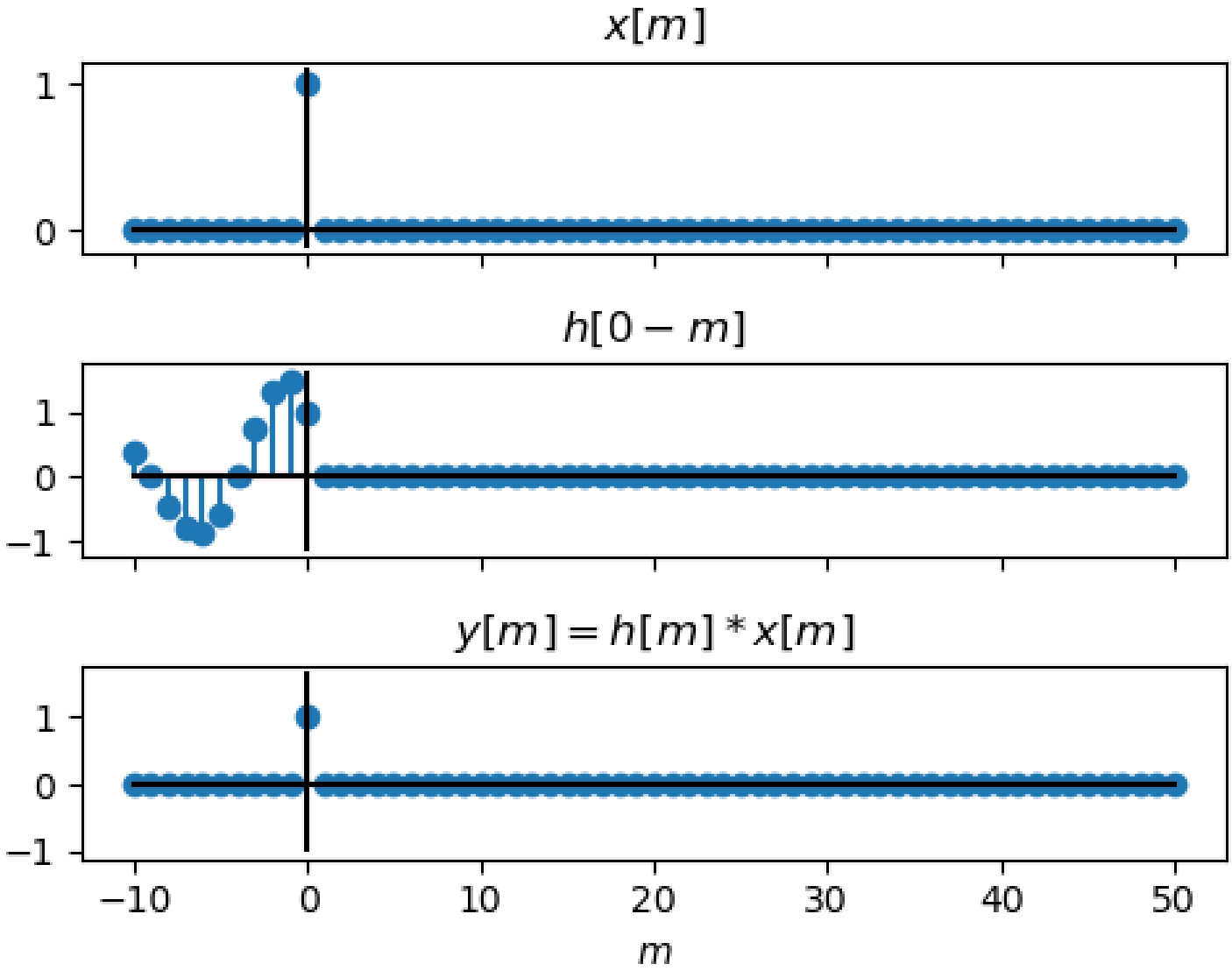
$$y[n] = x[n] + 2e^{-\sigma_1}\cos(\omega_1)y[n-1] - e^{-2\sigma_1}y[n-2]$$

Impulse response of a damped resonator

Putting $p_1 = e^{-\sigma_1 + j\omega_1}$ into the general form, we find that the impulse response of this filter is

$$h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1(n+1))u[n]$$

This is called a "stable resonator" or a "stable sinusoid" or a "damped resonator" or a "damped sinusoid." It rings at the frequency ω_1, but it gradually decays away.


$x[m]$
1
0
$h[0-m]$
1
0
−1
$y[m]=h[m]*x[m]$
1
0
−1
−10
0
10
20
30
40
50
m

A damped resonator is stable

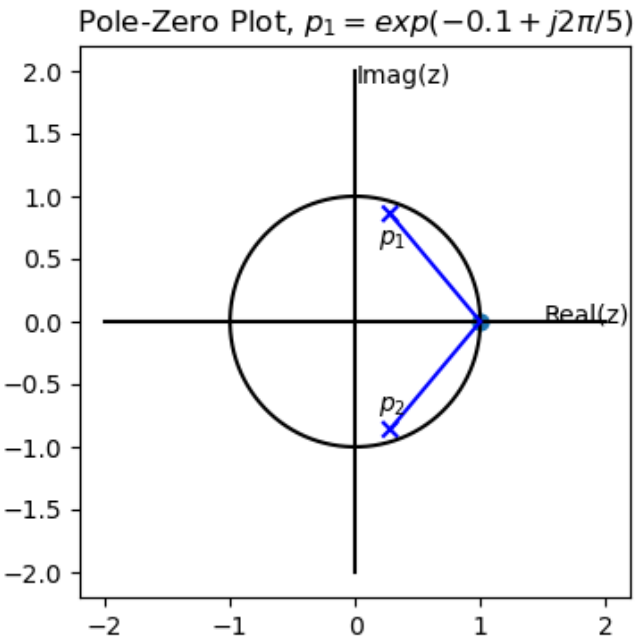
A damped resonator is stable: any finite input will generate a finite output.

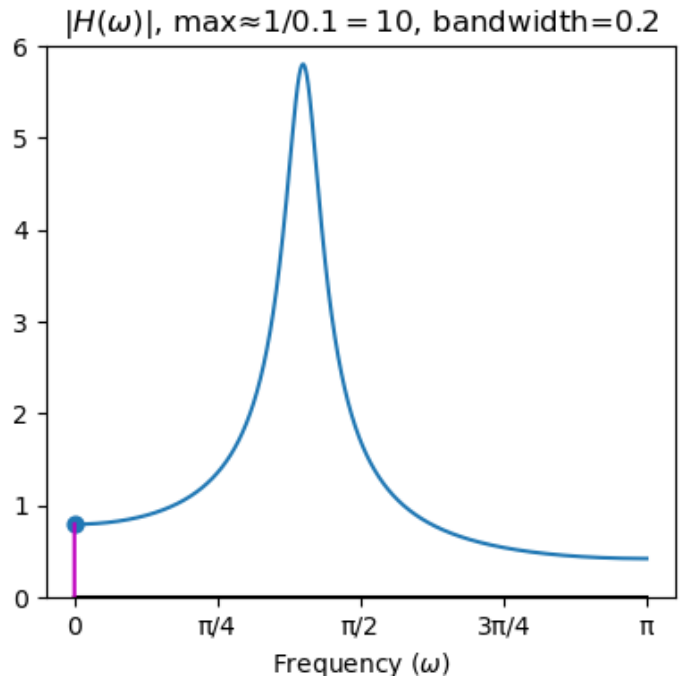
$$H(\omega) = H(z)|_{z=e^{j\omega}} = \frac{1}{(1 - e^{-\sigma_1 + j(\omega_1 - \omega)})(1 - e^{-\sigma_1 + j(-\omega_1 - \omega)})}$$

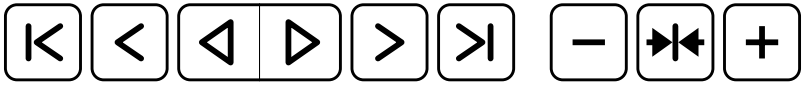
$$H(\omega_1) = \frac{1}{(1 - e^{-\sigma_1})(1 - e^{-\sigma_1 - 2j\omega_1})} \approx \frac{1}{1 - e^{-\sigma_1}} \approx \frac{1}{\sigma_1}$$

So if $x[n] = \cos(\omega_1 n)$, then $y[n]$ is

$$\begin{aligned} y[n] &= |H(\omega_1)| \cos\left(\omega_1 n + \angle H(\omega_1)\right) \\ &\approx \frac{1}{\sigma_1} \cos\left(\omega_1 n + \angle H(\omega_1)\right) \end{aligned}$$


Pole-Zero Plot, $p_1 = exp(-0.1 + j2\pi/5)$
Imag(z)
Real(z)
p_1
p_2
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2


$|H(\omega)|$, max≈1/0.1 = 10, bandwidth=0.2
6
5
4
3
2
1
0
0
π/4
π/2
3π/4
π
Frequency (ω)



Stability from the POV of the Impulse Response

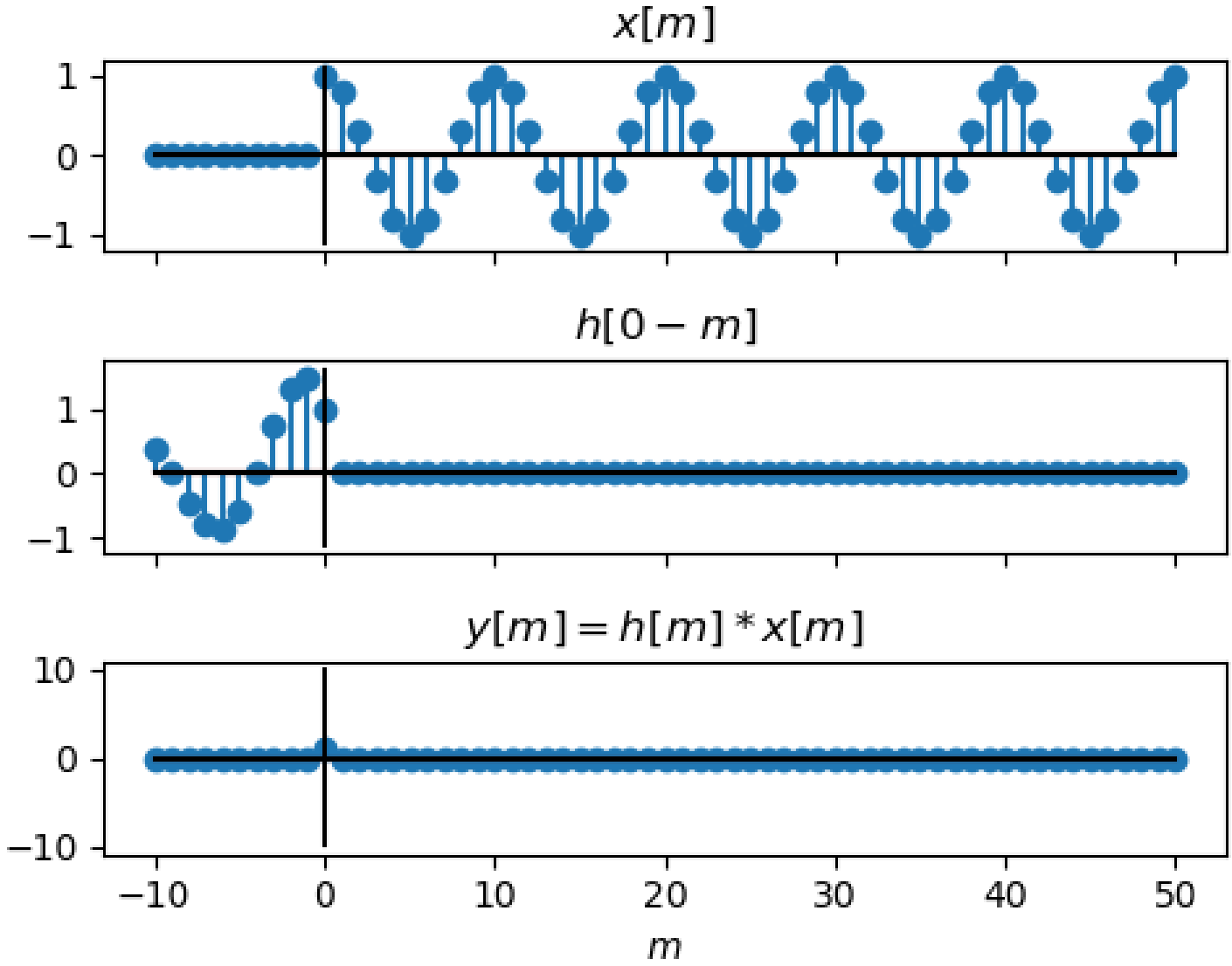
From the point of view of the impulse response, you can think of stability like this:

$$y[n] = \sum_m x[m]h[n-m]$$

Suppose $x[m] = \cos(\omega_1 m)u[m]$. Then

$$y[n] = x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \ldots$$

We keep adding extra copies of $h[n-m]$, for each m, forever. However, since each $h[n-m]$ dies away, and since they are being added with a time delay between them, the result never builds all the way to infinity.


$x[m]$
1
0
−1
$h[0-m]$
1
0
−1
$y[m]=h[m]*x[m]$
10
0
−10
−10
0
10
20
30
40
50
m

Outline

1. Review: Lossless resonators
2. Impulse response of a damped resonator
3. Bandwidth
4. Underdamped, Overdamped, and Critically-damped filters
5. Summary

Magnitude Response of an All-Pole Filter

For another useful visualization, let's re-write $|H(z)|$ as the product of two vector distances:

$$|H(z)| = \frac{1}{|1 - p_1 z^{-1}| \times |1 - p_2 z^{-1}|}$$

Magnitude Response at $\omega = \omega_1 \pm \epsilon$

Now let's suppose $p_1 = e^{-\sigma_1 + j\omega_1}$, and $p_2 = p_1^* = e^{-\sigma_1 - j\omega_1}$. Consider what happens when $\omega = \omega_1 \pm \epsilon$ for small values of ϵ.

$$|1 - p_1 z^{-1}| = \left|1 - e^{-\sigma_1 + j\omega_1 - j\omega_1 \pm j\epsilon}\right| = \left|1 - e^{-\sigma_1 \pm j\epsilon}\right|$$
$$|1 - p_2 z^{-1}| = \left|1 - e^{-\sigma_1 - j\omega_1 - j\omega_1 \pm j\epsilon}\right| \approx \text{constant}$$

The fourth line above assumes that $2\omega_1$ is much larger than either σ_1 or ϵ, so $\left|1 - e^{-\sigma_1 - j\omega_1 - j\omega_1 \pm j\epsilon}\right| \approx \left|1 - e^{-2j\omega_1}\right|$, which is a constant, independent of ϵ.

$$|H(\omega_1 \pm \epsilon)| = \frac{1}{|1 - p_1 z^{-1}| \times |1 - p_2 z^{-1}|}$$
$$\approx \frac{1/\text{constant}}{\left|1 - e^{-\sigma_1 \pm j\epsilon}\right|}$$

Magnitude Response at $\omega = \omega_1 \pm \epsilon$

Let's use the approximation $e^x \approx 1 + x$, which is true for small values of x. That gives us

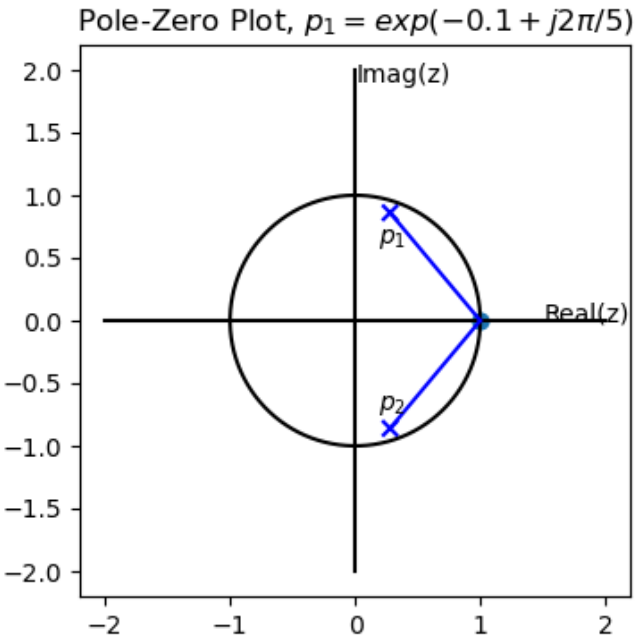
$$|H(\omega)| \approx \frac{1/\text{constant}}{|1 - e^{-\sigma_1 \pm j\epsilon}|} \approx \frac{1/\text{constant}}{|-\sigma_1 \pm j\epsilon|}$$

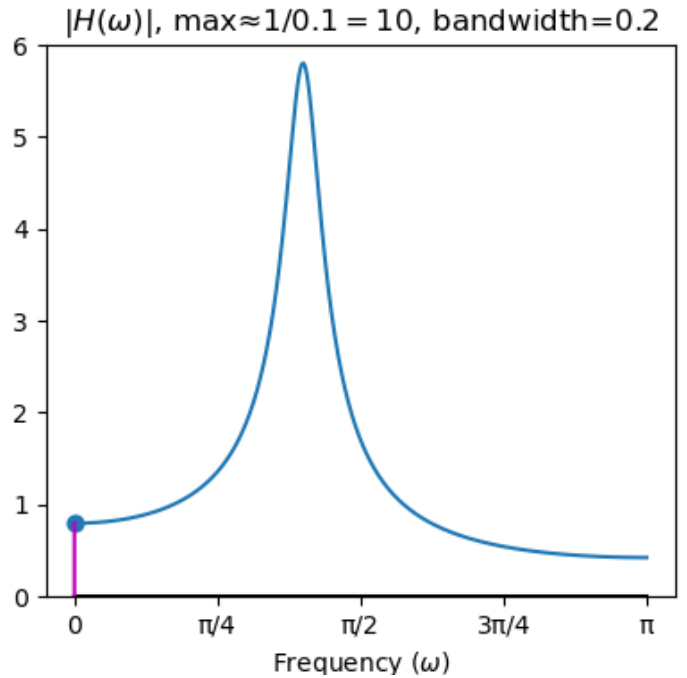
There are three frequencies that really matter:

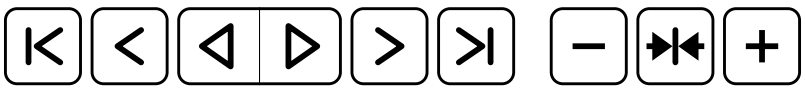
1. Right at the pole, at $\epsilon = 0$, we have $|H(\omega_1)| \approx \frac{1/\text{constant}}{\sigma_1}$
2. At $\pm$ half a bandwidth, $\omega = \omega_1 \pm \sigma_1$, we have

$$|H(\omega_1 \pm \sigma_1)| \approx \frac{1/\text{constant}}{|-\sigma_1 \pm j\sigma_1|} = \frac{1/\text{constant}}{\sigma_1\sqrt{2}} = \frac{1}{\sqrt{2}}|H(\omega_1)|$$

$|H(\omega_1 \pm \sigma_1)| = \frac{1}{\sqrt{2}}|H(\omega_1)|$







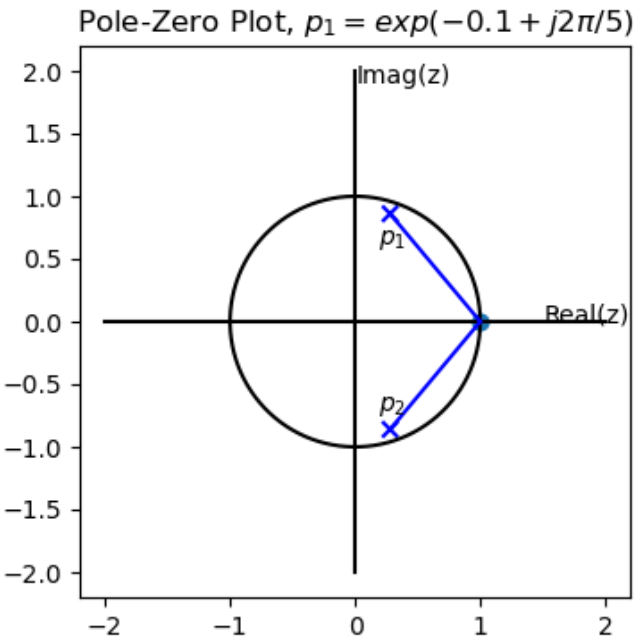
The 3dB-bandwidth is $2\sigma_1$ radians/sample

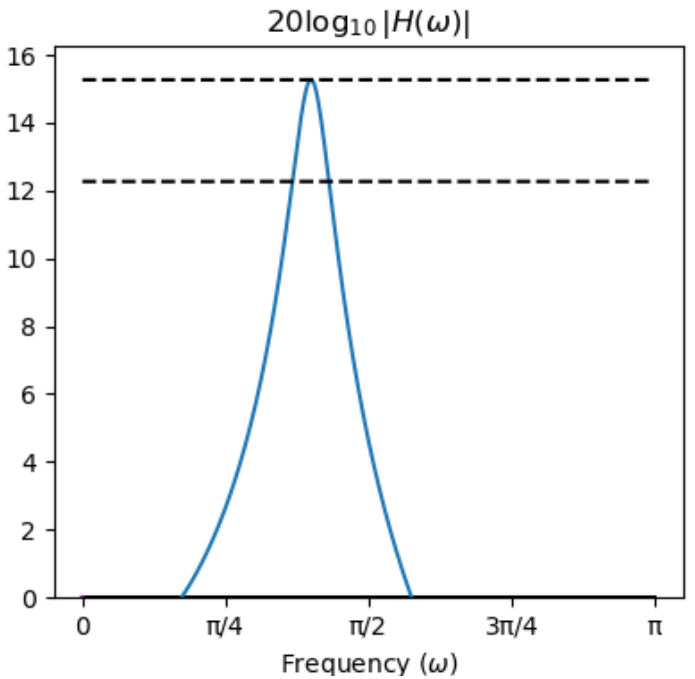
At $\omega = \omega_1 \pm \sigma_1$, we have

$$
\begin{aligned}
|H(\omega_1 \pm \sigma_1)| &= \frac{1}{\sqrt{2}} |H(\omega_1)| \\
20 \log_{10} |H(\omega_1 \pm \sigma_1)| &= 20 \log_{10} |H(\omega_1)| - 20 \log_{10} \sqrt{2} \\
&\approx 20 \log_{10} |H(\omega_1)| - 3\text{dB}
\end{aligned}
$$

- The "3dB bandwidth" of the filter is defined to be the width of the peak, at a height 3dB down from its peak.
- The 3dB bandwidth is the difference between $\omega_1 + \sigma_1$ and $\omega_1 - \sigma_1$, which is $2\sigma_1$.

3dB Bandwidth = width of the peak at a level of 3dB below the top





Outline

1. Review: Lossless resonators
2. Impulse response of a damped resonator
3. Bandwidth
4. Underdamped, Overdamped, and Critically-damped filters
5. Summary

A General Second-Order All-Pole Filter

Remember our general second-order all-pole system:

$$y[n] = x[n] - by[n-1] - cy[n-2]$$

$$p_1, p_2 = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - 4c}$$

There are three types of second-order filters:

1. **Underdamped filter:** $\left(\frac{b}{2}\right)^2 - 4c < 0$
2. **Overdamped filter:** $\left(\frac{b}{2}\right)^2 - 4c > 0$
3. **Critically-damped filter:** $\left(\frac{b}{2}\right)^2 - 4c = 0$

1. Underdamped filter

An **underdamped filter** is one for which $\left(\frac{b}{2}\right)^2 - 4c < 0$. In this case both $p_1 = e^{-\sigma_1 + j\omega_1}$ and $p_2 = e^{-\sigma_1 - j\omega_1}$ are complex:

$$\begin{aligned} p_1, p_2 &= -\frac{b}{2} \pm j\sqrt{4c - \left(\frac{b}{2}\right)^2} \\ &= -\sigma_1 \pm j\omega_1 \end{aligned}$$

and the impulse response is

$$h[n] = \frac{e^{-\sigma_1(n+1)} \sin(\omega_1(n+1))}{e^{-\sigma_1} \sin(\omega_1)} u[n]$$

We call such filters "under-damped" because the damping is not enough to keep them from resonating.

2. Overdamped filter

An **overdamped filter** is one for which $\left(\frac{b}{2}\right)^2 - 4c > 0$. In this case, both p_1 and p_2 are real:

$$p_1, p_2 = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - 4c}$$

and the impulse response is

$$h[n] = \left(\frac{p_1^{n+1}}{p_1 - p_2} + \frac{p_2^{n+1}}{p_2 - p_1} \right) u[n]$$

Such filters are "over-damped" because the damping prevents resonance. It's sometimes useful to talk about two different "decay rates," $\sigma_1 = -\ln(p_1)$ and $\sigma_2 = -\ln(p_2)$, one of which is faster than the other.

3. Critically-damped filter

A **critically damped** filter is one for which $\left(\frac{b}{2}\right)^2 - 4c = 0$. In this case

$$p_1 = p_2 = -\frac{b}{2}$$

$$H(z) = \left(\frac{1}{1 - p_1 z^{-1}}\right)^2$$

$$h[n] = (p_1^n u[n]) * (p_1^n u[n])$$

We call such filters "critically damped" because they are damped just barely enough to prevent resonance. We can define its "decay rate" to be $\sigma = -\ln(p_1)$.

Try the quiz!

Go to the course web page, and try the quiz!

Outline

1. Review: Lossless resonators
2. Impulse response of a damped resonator
3. Bandwidth
4. Underdamped, Overdamped, and Critically-damped filters
5. Summary

Impulse Response of a Second-Order All-Pole Filter

A general all-pole filter has the system function

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} = \frac{1}{1 - (p_1 + p_1^*)z^{-1} + p_1 p_1^* z^{-2}}$$

Its impulse response is

$$h[n] = g_1 p_1^n u[n] + g_1^* (p_1^*)^n u[n]$$

Impulse Response of a Second-Order All-Pole Filter

We can take advantage of complex numbers to write these as

$$H(z) = \frac{1}{1 - 2e^{-\sigma_1}\cos(\omega_1)z^{-1} + e^{-2\sigma_1}z^{-2}}$$

and

$$h[n] = \frac{1}{\sin(\omega_1)}e^{-\sigma_1 n}\sin(\omega_1(n+1))u[n]$$

Magnitude Response of a Second-Order All-Pole Filter

In the frequency response, there are three frequencies that really matter:

1. Right at the pole, at $\omega = \omega_1$, we have

$$|H(\omega_1)| \propto \frac{1}{\sigma_1}$$

2. At $\pm$ half a bandwidth, $\omega = \omega_1 \pm \sigma_1$, we have

$$|H(\omega_1 \pm \sigma_1)| = \frac{1}{\sqrt{2}}|H(\omega_1)|$$

Underdamped, Overdamped, and Critically-damped filters

There are three types of second-order filters:

1. **Underdamped filter:** $\left(\frac{b}{2}\right)^2 - 4c < 0$

$$h[n] = \frac{e^{-\sigma_1(n+1)} \sin(\omega_1(n+1))}{e^{-\sigma_1} \sin(\omega_1)} u[n]$$

2. **Overdamped filter:** $\left(\frac{b}{2}\right)^2 - 4c > 0$

$$h[n] = \left(\frac{p_1^{n+1}}{p_1 - p_2} + \frac{p_2^{n+1}}{p_2 - p_1} \right) u[n]$$

3. **Critically-damped filter:** $\left(\frac{b}{2}\right)^2 - 4c = 0$

$$h[n] = (p_1^n u[n]) * (p_1^n u[n])$$