# Lecture 30: Lossless Resonator

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ECE 401: Signal and Image Analysis

1. Review: Poles and Zeros
2. Impulse Response of a Second-Order Filter
3. Lossless Resonator
4. Summary

# Outline

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# Review: Poles and Zeros

A first-order autoregressive filter,

$$y[n] = x[n] - bx[n-1] + ay[n-1],$$

has the impulse response and transfer function

$$h[n] = a^n u[n] - ba^{n-1} u[n-1] \leftrightarrow H(z) = \frac{1 - bz^{-1}}{1 - az^{-1}},$$

where $a$ is called the **pole** of the filter, and $b$ is called its **zero**.

# Causality and Stability

- A filter is **causal** if and only if the output, $y[n]$, depends only an **current and past** values of the input, $x[n], x[n-1], x[n-2], \ldots$.
- A filter is **stable** if and only if **every** finite-valued input generates a finite-valued output. A causal first-order IIR filter is stable if and only if $|a| < 1$.

# Review: Poles and Zeros

Suppose $H(z) = \frac{1-bz^{-1}}{1-az^{-1}}$, and $|a| < 1$. Now let's evaluate $|H(\omega)|$, by evaluating $|H(z)|$ at $z = e^{j\omega}$:

$$|H(\omega)| = \frac{|e^{j\omega} - b|}{|e^{j\omega} - a|}$$

What it means $|H(\omega)|$ is the ratio of two vector lengths:

- When the vector length $|e^{j\omega} - b|$ is small, then $|H(\omega)|$ is small.
- When $|e^{j\omega} - a|$ is small, then $|H(\omega)|$ is LARGE.

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# A General Second-Order All-Pole Filter

The next few lectures will study a filter with one input, and with two feedback terms:

$$y[n] = x[n] - by[n-1] - cy[n-2]$$
$$Y(z) = X(z) - (bz^{-1} + cz^{-2})Y(z)$$
$$H(z) = \frac{1}{1 + bz^{-1} + cz^{-2}}$$

Using the quadratic formula, we can factor the denominator as

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$
$$p_1, p_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Most interesting cases will have $4c > b^2$, so $p_1$ will be complex-valued, and $p_2 = p_1^*$.

# What's its impulse response?

What is the impulse response of this filter? We have many different ways to find out:

1. **Difference equation:** Put $x[n] = \delta[n]$ into the system, see what comes out.
2. **Inverse DTFT:** Set $z = e^{j\omega}$ to find $H(\omega)$, then take its inverse DTFT.
3. **Inverse Z transform:** Since we already know the inverse transform of a first-order filter, we could factor $H(z)$ into...
   - the product of two first-order filters, or...
   - the sum of two first-order filters

# 1. Difference Equation

Plug $x[n] = \delta[n]$ into the difference equation:

$$h[n] = \delta[n] - bh[n-1] - ch[n-2]$$

We get:

$$\begin{aligned} h[0] &= 1 \\ h[1] &= -b \\ h[2] &= b^2 - c \\ &\vdots \end{aligned}$$

- Using a computer, we could generate $h[n]$ iteratively this way. $h[n]$ is infinite length, so we'd never generate all of it, but we could generate a few thousand samples.
- Doing this iteratively doesn't really help us to understand it, though. Can we do something that's a bit more human?

# 2. Inverse DTFT

If we set $z = e^{j\omega}$, we get

$$H(\omega) = \frac{1}{1 + be^{-j\omega} + ce^{-2j\omega}}$$

- Using a computer, we could generate samples of $H(\omega)$ for a few thousand samples of $\omega$, then use `np.fft.ifft` to compute the inverse DFT, which would be a finite-length approximation of $h[n]$.
- Doing this with `np.fft.ifft` doesn't really help us to understand it, though. Can we do something that's a bit more human?

# 3a. Inverse Z transform: Product of two filters

Notice that $H(z)$ can be factored as $H(z) = H_1(z)H_2(z)$, where

$$H_1(z) = \frac{1}{1 - p_1 z^{-1}}, \quad H_2(z) = \frac{1}{1 - p_1^* z^{-1}}$$

We already know the inverse Z transforms of these two filters. They're both one-pole feedback filters,

$$h_1[n] = p_1^n u[n], \quad h_2[n] = (p_1^*)^n u[n]$$

Since $H(z) = H_1(z)H_2(z)$, therefore

$$h[n] = h_1[n] * h_2[n]$$

This is a convolution that you know how to do. We could solve the problem this way. But it turns out that there is a different solution method that, though it involves a few more steps, gives us much more useful intuitions.

# 3b. Inverse Z transform: Sum of two filters

Suppose that, instead of multiplying $H_1(z)$ and $H_2(z)$, we add them:

$$g_1 H_1(z) + g_2 H_2(z) = \frac{g_1}{1 - p_1 z^{-1}} + \frac{g_2}{1 - p_1^* z^{-1}}$$
$$= \frac{g_1(1 - p_1^* z^{-1}) + g_2(1 - p_1 z^{-1})}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})}$$

Suppose we want to make $H(z) = g_1 H_1(z) + g_2 H_2(z)$. We can do that by choosing $g_1$ and $g_2$ so that:

$$\frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} = \frac{g_1(1 - p_1^* z^{-1}) + g_2(1 - p_1 z^{-1})}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})}$$

Solving, we get

$$g_1 = \frac{p_1}{p_1 - p_1^*}, \quad g_2 = g_1^*$$

# Impulse Response of a Second-Order IIR

So if we choose

$$g_1 = \frac{p_1}{p_1 - p_1^*}, \quad g_2 = g_1^*$$

. . . then we get. . .

$$H(z) = g_1 H_1(z) + g_2 H_2(z)$$
$$h[n] = g_1 h_1[n] + g_2 h_2[n]$$

Plugging in $h_1[n] = p_1^n u[n]$ and $h_2[n] = (p_1^*)^n u[n]$, we get

$$h[n] = \frac{p_1^{n+1} - (p_1^*)^{n+1}}{p_1 - p_1^*} u[n]$$

# Impulse Response of a Second-Order IIR

$$
\begin{aligned}
H(z) &= \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} \\
h[n] &= g_1 h_1[n] + g_2 h_2[n] \\
&= \frac{p_1^{n+1} - (p_1^{n+1})^*}{p_1 - p_1^*} u[n]
\end{aligned}
$$

That looks like it might be complex-valued, but it isn't. Noticing that $p_1 - p_1^* = 2j\Im(p_1)$, the imaginary part of $p_1$, we can write $h[n]$ as the following real-valued function of time:

$$
h[n] = \frac{\Im\left(p_1^{n+1}\right)}{\Im\left(p_1\right)} u[n]
$$

# Outline

1 Review: Poles and Zeros

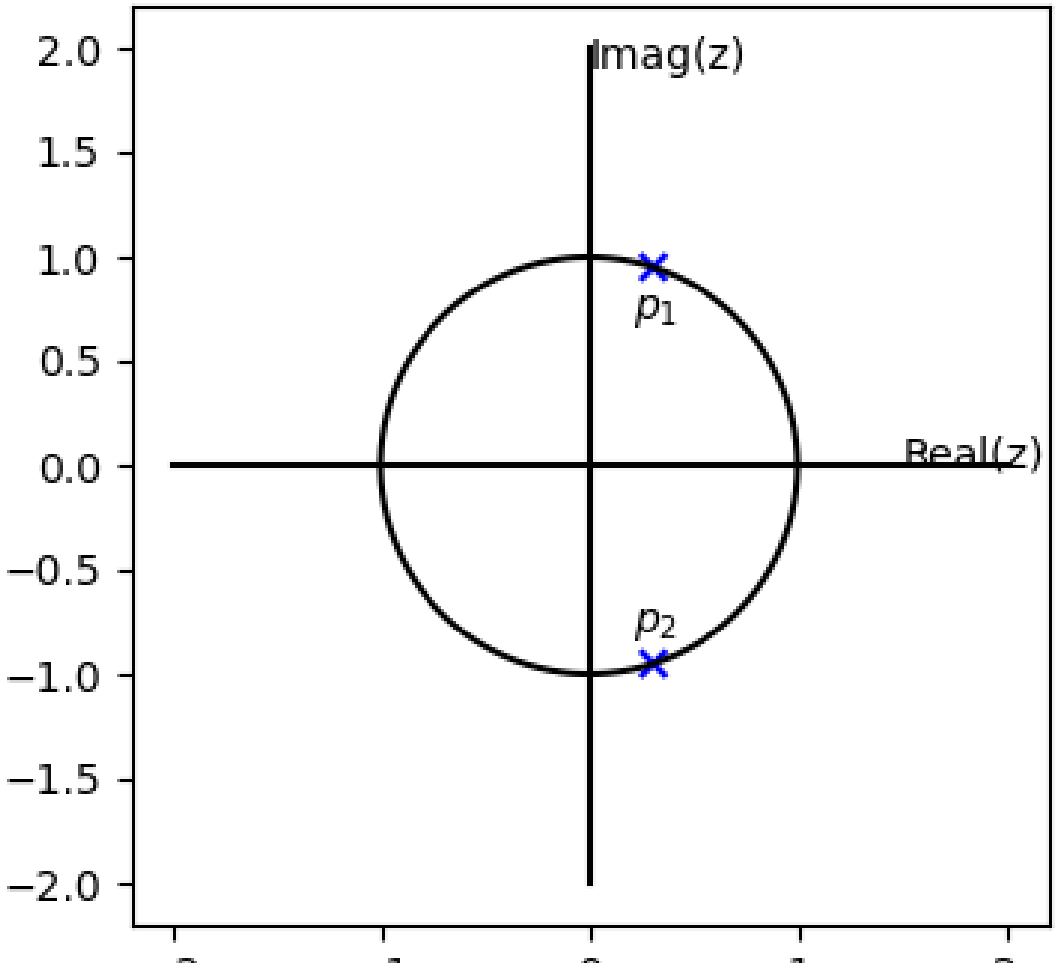
2 Impulse Response of a Second-Order Filter

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# Example: Ideal resonator

As the first example, let's suppose we put $p_1$ right on the unit circle, $p_1 = e^{j\omega_1}$.



# Transfer function of an ideal resonator

The transfer function of an ideal resonator is

$$H(z) = \frac{1}{(1 - e^{-j\omega_1} z^{-1})(1 - e^{j\omega_1} z^{-1})}$$
$$= \frac{1}{1 - 2\cos(\omega_1) z^{-1} + z^{-2}}$$

# Impulse response of an ideal resonator

Using the sum-of-first-order-sections method, we have:

$$g_1 = \frac{e^{j\omega_1}}{e^{j\omega_1} - e^{-j\omega_1}} = \frac{e^{j\omega_1}}{2j\sin(\omega_1)}$$

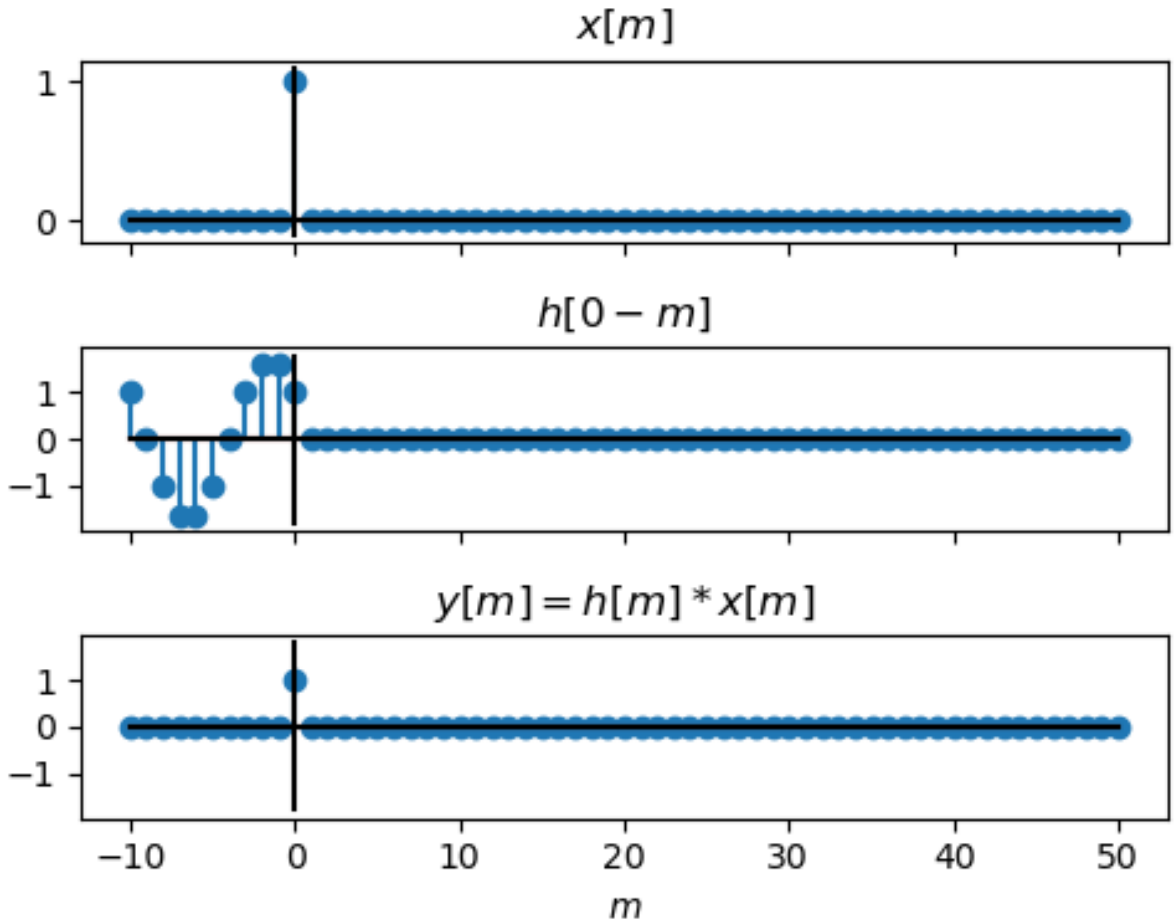
$$g_2 = g_1^* = \frac{e^{-j\omega_1}}{-2j\sin(\omega_1)}$$

Since $H(z) = g_1 H_1(z) + g_2 H_2(z)$,

$$\begin{aligned} h[n] &= g_1 h_1[n] + g_2 h_2[n] \\ &= \frac{e^{j\omega_1(n+1)} - e^{-j\omega_1(n+1)}}{2j\sin(\omega_1)} u[n] \\ &= \frac{\sin(\omega_1(n+1))}{\sin(\omega_1)} u[n] \end{aligned}$$

Notice that $h[n]$ is real-valued, because we chose $p_2 = p_1^*$ and $g_2 = g_1^*$.

This is called an "ideal resonator" because it keeps ringing forever, as a perfect sine wave, without ever decaying away.



# An Ideal Resonator is Unstable

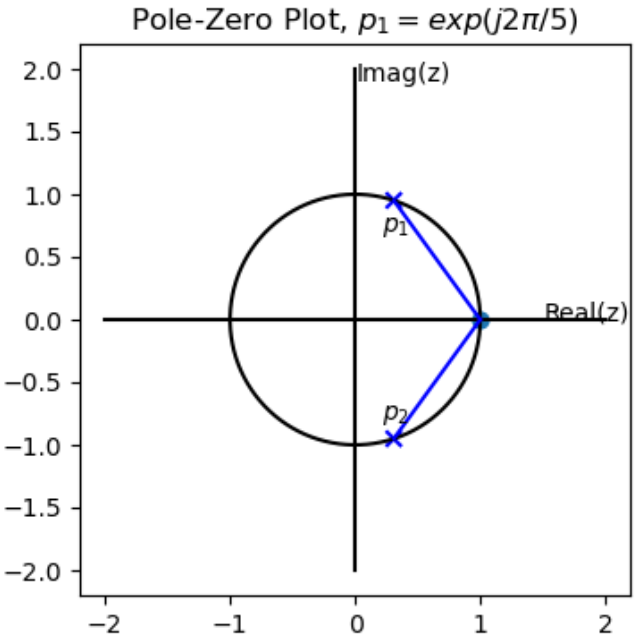
An ideal resonator is unstable. The easiest way to see what this means is by looking at its frequency response:

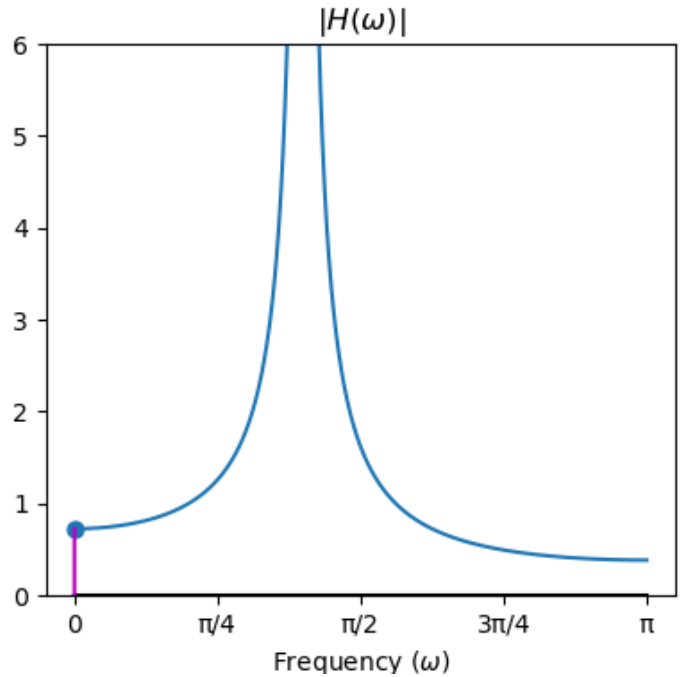
$$H(\omega) = H(z)|_{z=e^{j\omega}} = \frac{1}{(1 - e^{j(\omega_1 - \omega)})(1 - e^{j(-\omega_1 - \omega)})}$$

$$H(\omega_1) = \frac{1}{(1-1)(1 - e^{-2j\omega_1})} = \infty$$

So if $x[n] = \cos(\omega_1 n)$, then $y[n]$ is

$$y[n] = |H(\omega_1)| \cos\left(\omega_1 n + \angle H(\omega_1)\right) = \infty$$


Pole-Zero Plot, $p_1 = exp(j2\pi/5)$
Imag(z)
Real(z)
$p_1$
$p_2$
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2


$|H(\omega)|$
6
5
4
3
2
1
0
0
π/4
π/2
3π/4
π
Frequency ($\omega$)

# Instability from the POV of the Impulse Response

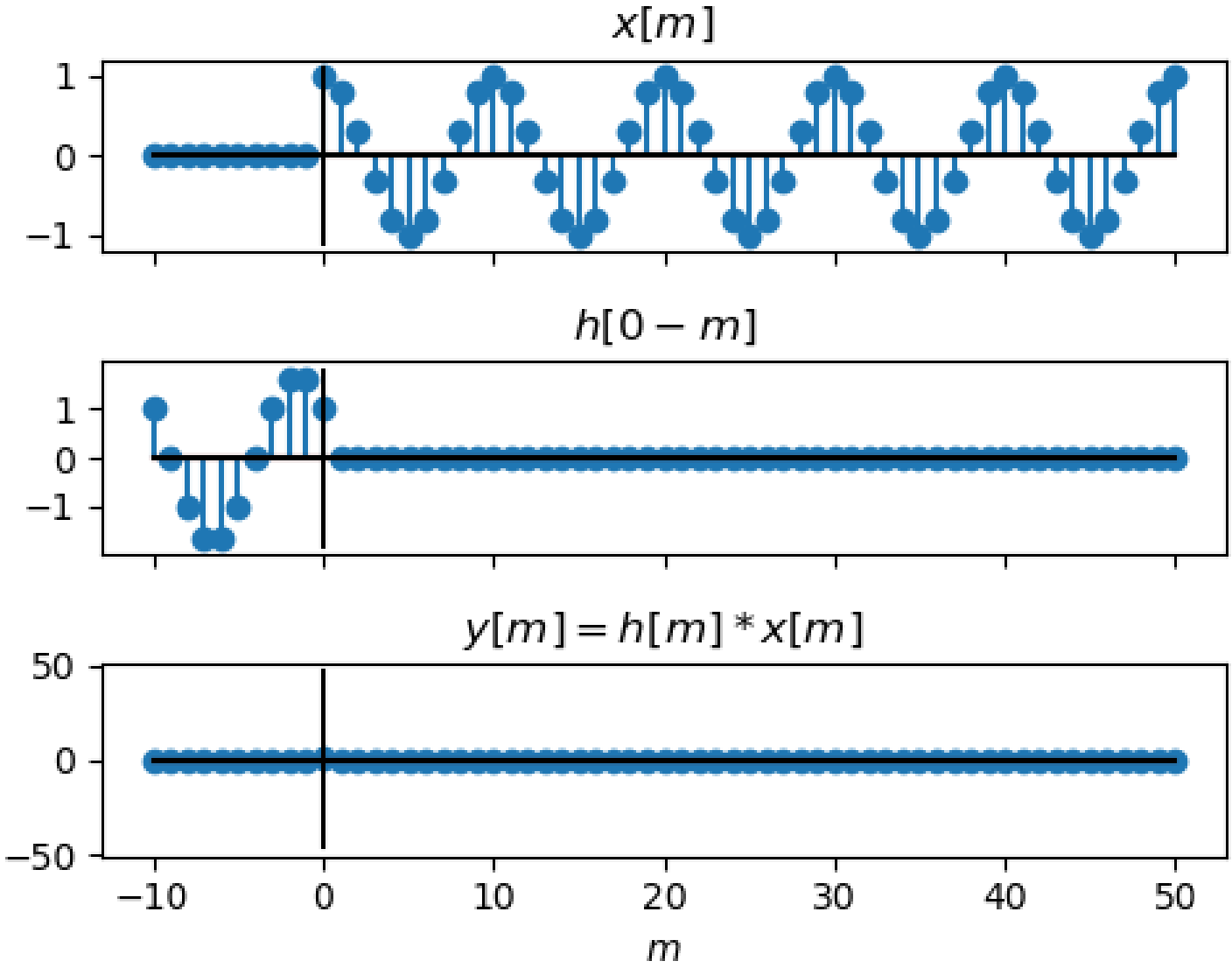
From the point of view of the impulse response, you can think of instability like this:

$$y[n] = \sum_m x[m]h[n-m]$$

Suppose $x[m] = \cos(\omega_1 m)u[m]$. Then

$$y[n] = x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \ldots$$

We keep adding extra copies of $h[n-m]$, for each $m$, forever. Since $h[n]$ never dies away, the result is that we keep building up $y[n]$ toward infinity.


$x[m]$
1
0
−1
$h[0-m]$
1
0
−1
$y[m]=h[m]*x[m]$
50
0
−50
−10
0
10
20
30
40
50
$m$

# Try the quiz!

Try the quiz!

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# Impulse Response of a Second-Order All-Pole Filter

A general all-pole filter has the system function

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} = \frac{1}{1 - (p_1 + p_1^*)z^{-1} + p_1 p_1^* z^{-2}}$$

Its impulse response is

$$\begin{aligned} h[n] &= g_1 p_1^n u[n] + g_1^* (p_1^*)^n u[n] \\ &= \frac{\Im(p_1^{n+1})}{\Im(p_1)} u[n] \end{aligned}$$

# Impulse Response of a Lossless Resonator

A lossless resonator has its poles at $p_1 = e^{j\omega_1}$ and $p_2 = e^{-j\omega_1}$, so

$$H(z) = \frac{1}{1 - 2\cos(\omega)z^{-1} + z^{-2}}$$

Its impulse response is

$$\begin{aligned} h[n] &= g_1 p_1^n u[n] + g_1^* (p_1^*)^n u[n] \\ &= \frac{\sin(\omega_1(n+1))}{\sin(\omega_1)} u[n] \end{aligned}$$