Lecture 29: Poles and Zeros

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ECE 401: Signal and Image Analysis

1. Review: Ideal Filters, DTFT, and the Z Transform
2. Finding the Zeros of $H(z)$
3. Finding the Poles and Zeros of $H(z)$
4. Summary

Outline

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Review: Frequency Response

- **Tones in $\rightarrow$ Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)|\cos(\omega n + \angle H(\omega))$$
$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|H(\omega)|\cos(\omega n + \theta + \angle H(\omega))$$

- where the **Frequency Response** is given by

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

Ideal Filters

Ideal Lowpass Filter:

$$H_{LPF}(\omega) = \begin{cases} 1 & |\omega| \leq \omega_L, \\ 0 & \omega_c < |\omega| \leq \pi. \end{cases} \quad \leftrightarrow \quad h_{LPF}[m] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n)$$

Ideal Filters

Ideal Bandpass Filter:

$$H_{BPF}(\omega) = H_{LPF}(\omega|\omega_2) - H_{LPF}(\omega|\omega_1)$$

$$\leftrightarrow \quad h_{BPF}[n] = \frac{\omega_2}{\pi}\text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi}\text{sinc}(\omega_1 n)$$

Ideal Filters

Delayed Ideal Highpass Filter:

$$H_{HPF}(\omega) = e^{-j\omega n_0}\left(1 - H_{LPF}(\omega)\right)$$

$$\leftrightarrow$$

$$h_{HPF}[n] = \begin{cases} \delta[n-n_0] - \frac{\omega_c}{\pi}\text{sinc}(\omega_c(n-n_0)) & n_0 = \text{integer} \\ \text{sinc}(\pi(n-n_0)) - \frac{\omega_c}{\pi}\text{sinc}(\omega_c(n-n_0)) & \text{otherwise} \end{cases}$$

Summary: Z Transform

- A **difference equation** is an equation in terms of time-shifted copies of $x[n]$ and/or $y[n]$.
- We have less to write if we use a new frequency variable, $z = e^{j\omega}$. This leads us to the **Z transform**:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- We can find the **transfer function** $H(z) = Y(z)/X(z)$ by taking the Z transform of each term of the difference equation. This will result in the ratio of two polynomials in z.
- We can find the **frequency response** by setting $z = e^{j\omega}$.

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Treating $H(z)$ as a Polynomial

Suppose we have

$$h[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-2]$$

- Is this a low-pass filter, a high-pass filter, or something else?
- $H(z)$ provides a new way of thinking about the frequency response: not an ideal LPF or HPF or BPF, but instead, something with particular **zeros** in the frequency domain.

Treating $H(z)$ as a Polynomial

Here's the transfer function $H(z)$:

$$H(z) = 1 - 2z^{-1} + 2z^{-2}$$

Notice that we can factor that, just like any other polynomial:

$$H(z) = \frac{1}{z^2}\left(z^2 - 2z + 2\right)$$

Using the quadratic formula, we can find its roots:

$$z = \frac{2 \pm \sqrt{(2)^2 - 4 \times 2}}{2} = 1 \pm j$$

Treating $H(z)$ as a Polynomial

We've discovered that $H(z)$ can be written as a product of factors:

$$H(z) = 1 - 2z^{-1} + 2z^{-2} = \frac{1}{z^2}(z - z_1)(z - z_2),$$

where the roots of the polynomial are

$$z_1 = 1 + j = \sqrt{2}e^{j\pi/4}$$
$$z_2 = 1 - j = \sqrt{2}e^{-j\pi/4}$$

The Zeros of $H(z)$

- The roots, z_1 and z_2, are the values of z for which $H(z) = 0$.
- But what does that mean? We know that for $z = e^{j\omega}$, $H(z)$ is just the frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

but the roots do not have unit magnitude:

$$z_1 = 1 + j = \sqrt{2}e^{j\pi/4}$$
$$z_2 = 1 - j = \sqrt{2}e^{-j\pi/4}$$

- What it means is that, when $\omega = \frac{\pi}{4}$ (so $z = e^{j\pi/4}$), then $|H(\omega)|$ is as close to a zero as it can possibly get. So at that frequency, $|H(\omega)|$ is as low as it can get.

Vectors in the Complex Plane

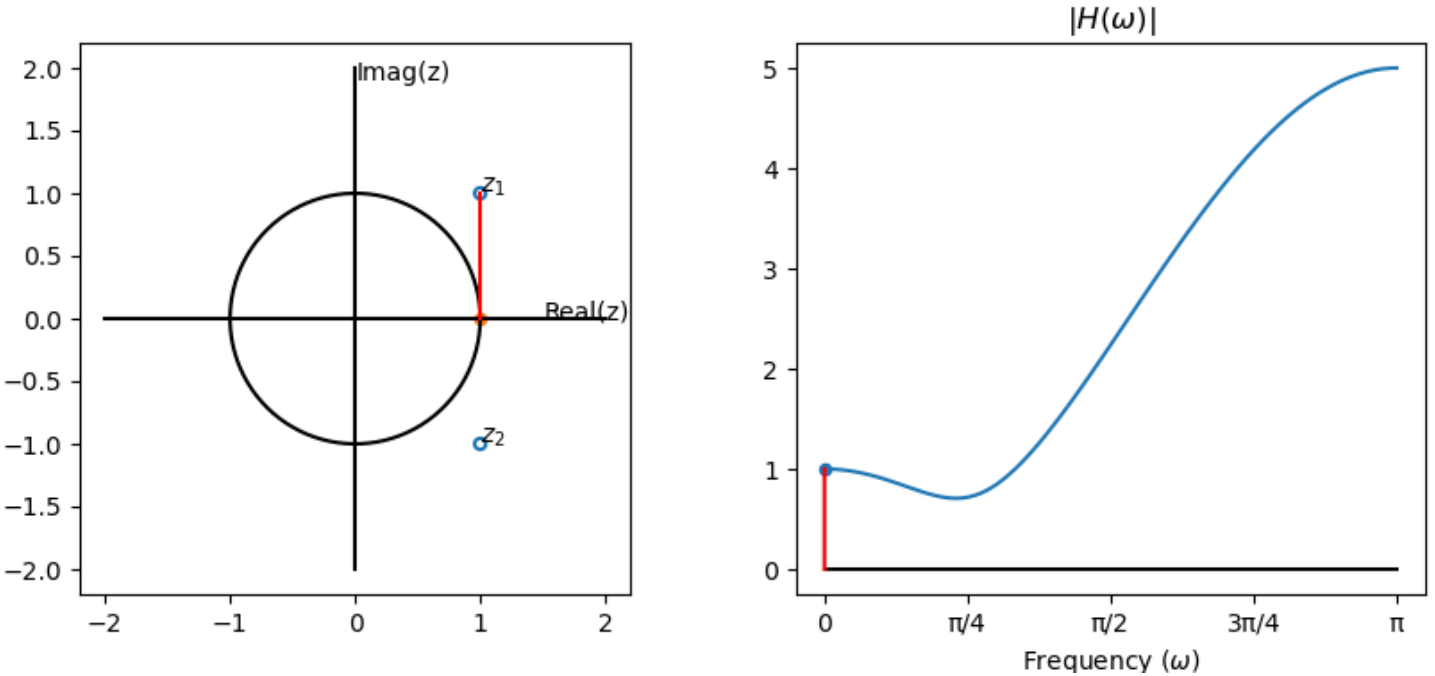
Suppose we write $|H(z)|$ like this:

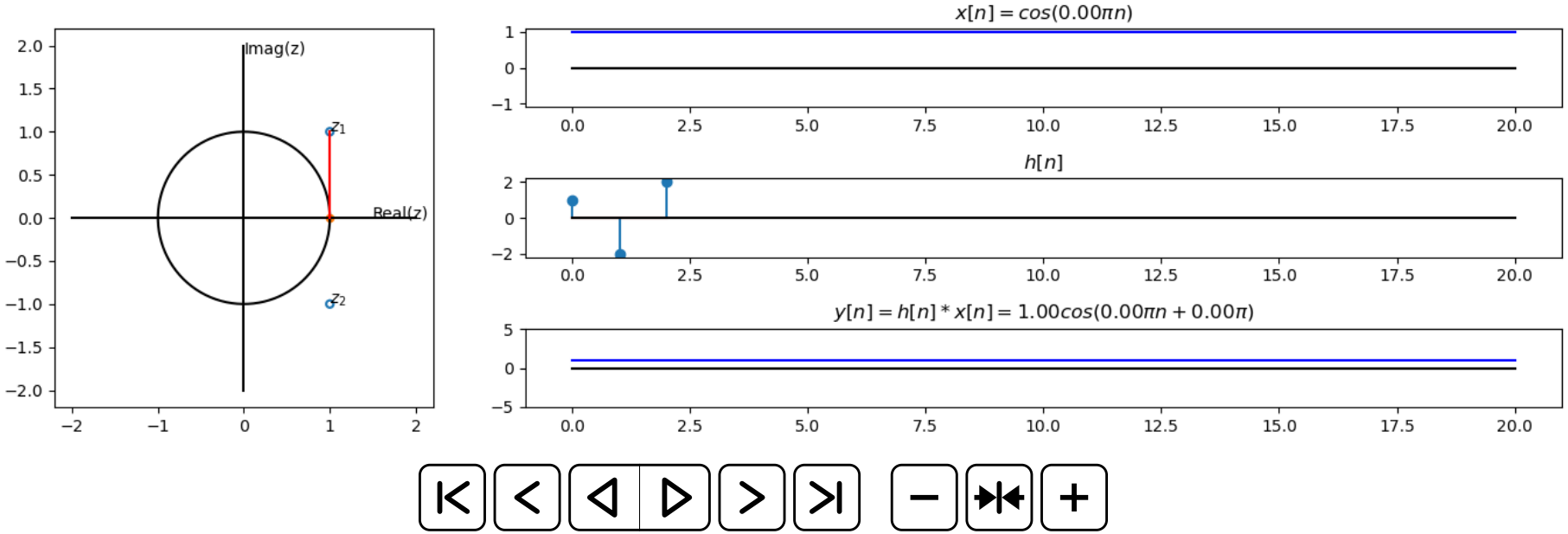
$$|H(z)| = \frac{1}{|z|^2} \times |z - z_1| \times |z - z_2| = |z - z_1| \times |z - z_2|$$

Now let's evaluate at $z = e^{j\omega}$:

$$|H(\omega)| = |e^{j\omega} - z_1| \times |e^{j\omega} - z_2|$$

What we've discovered is that $|H(\omega)|$ is small when the vector distance $|e^{j\omega} - z_1|$ is small, in other words, when $z = e^{j\omega}$ is as close as possible to one of the zeros.


Imag(z)
Real(z)
z_1
z_2
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
$|H(\omega)|$
5
4
3
2
1
0
0
π/4
π/2
3π/4
π
Frequency (ω)


Imag(z)
Real(z)
z_1
z_2
$x[n] = cos(0.00\pi n)$
$h[n]$
$y[n] = h[n] * x[n] = 1.00cos(0.00\pi n + 0.00\pi)$

Try the quiz!

Try the quiz!

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First-Order Filter

Now, let's find the frequency response of a general first-order filter, including BOTH feedforward and feedback delays:

$$y[n] = x[n] - bx[n-1] + ay[n-1],$$

where we'll assume that $|a| < 1$, so the filter is stable.

Frequency Response of a First-Order Filter

If $|a| < 1$, we can find the frequency response by taking the Z transform of each term in this equation:

$$y[n] = x[n] - bx[n-1] + ay[n-1],$$
$$Y(z) = X(z) - bz^{-1}X(z) + az^{-1}Y(z),$$

which we can solve to get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - bz^{-1}}{1 - az^{-1}}$$
$$H(\omega) = \frac{1 - be^{-j\omega}}{1 - ae^{-j\omega}}.$$

Treating $H(z)$ as a Ratio of Two Polynomials

Notice that $H(z)$ is the ratio of two polynomials:

$$H(z) = \frac{1 - bz^{-1}}{1 - az^{-1}} = \frac{z - b}{z - a}$$

- $z = b$ is called the **zero** of $H(z)$, because $H(z) = 0$ at $z = b$.
- $z = a$ is called the **pole** of $H(z)$, meaning that $H(z) \to \infty$ as $z \to a$.
- Remember z is just a shorthand for $z = e^{j\omega}$, so the conditions above can only happen if $|b| = 1$ or $|a| = 1$ ($b = e^{j\omega_b}$ or $a = e^{j\omega_a}$ for some frequencies ω_b and ω_a).

The Pole and Zero of $H(z)$

- The pole, $z = a$, and zero, $z = b$, are the values of z for which $H(z) = \infty$ and $H(z) = 0$, respectively.
- But what does that mean? We know that for $z = e^{j\omega}$, $H(z)$ is just the frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

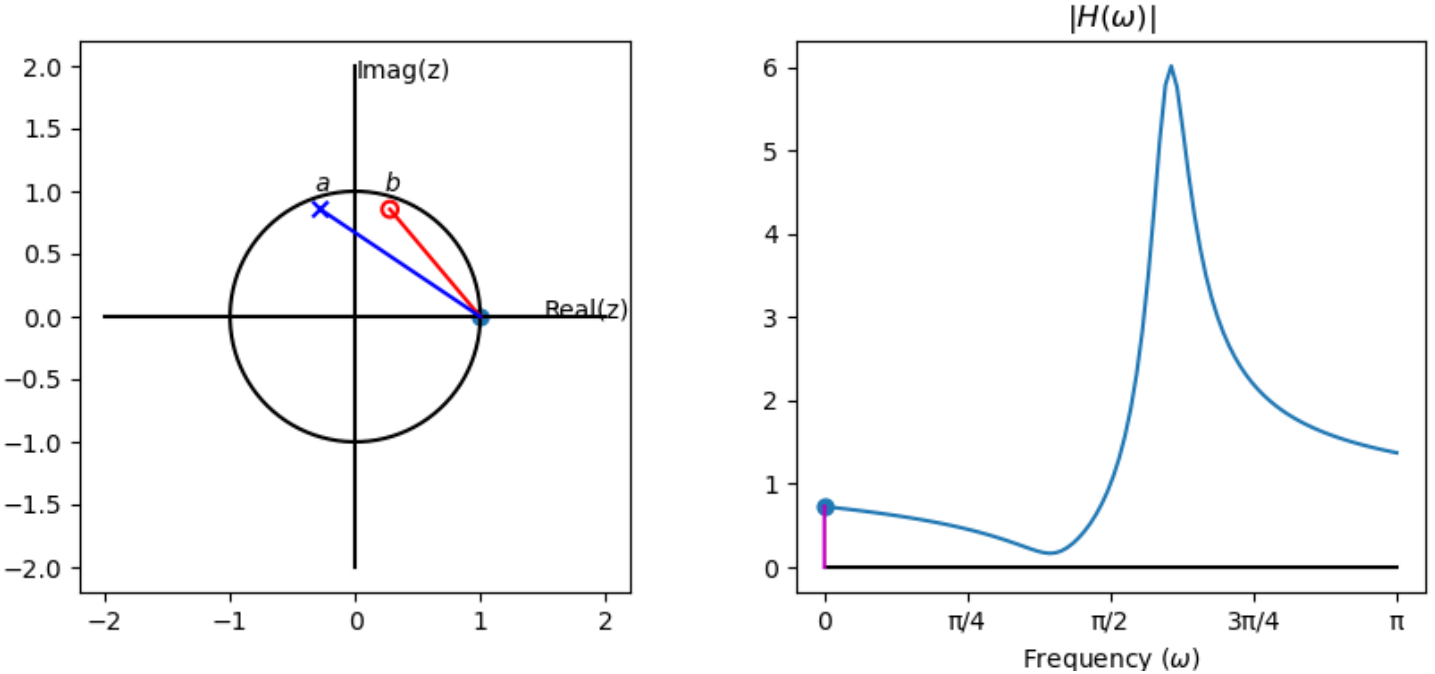
  but if the filter is stable, $|a| < 1$ and $z = e^{j\omega}$, so $z = a$ will never happen.
- What it means is that:
  - When $\omega = \angle b$, then $|H(\omega)|$ is as close to a zero as it can possibly get, so at that that frequency, $|H(\omega)|$ is as low as it can get.
  - When $\omega = \angle a$, then $|H(\omega)|$ is as close to a pole as it can possibly get, so at that that frequency, $|H(\omega)|$ is as high as it can get.

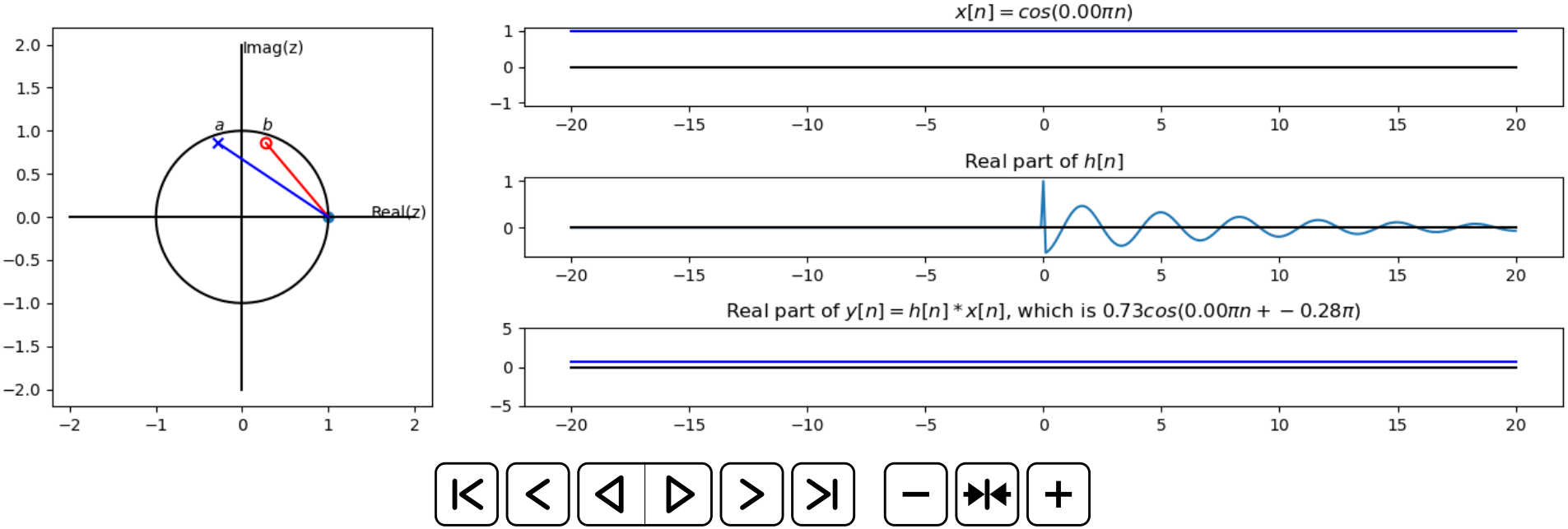
Vectors in the Complex Plane

Suppose we write $|H(z)|$ like this:

$$|H(z)| = \frac{|z-b|}{|z-a|}$$

What we've discovered is that $|H(z)|$ is small when the vector distance $|z-b| = |e^{j\omega} - b|$ is small, but LARGE when the vector distance $|z-a| = |e^{j\omega} - a|$ is small.


Imag(z)
Real(z)
a
b
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
|H(ω)|
6
5
4
3
2
1
0
0
π/4
π/2
3π/4
π
Frequency (ω)


Imag(z)
Real(z)
a
b
$x[n] = cos(0.00\pi n)$
Real part of $h[n]$
Real part of $y[n] = h[n] * x[n]$, which is $0.73cos(0.00\pi n + -0.28\pi)$

Why This is Useful

Now we have another way of thinking about frequency response.

- Instead of just LPF, HPF, or BPF, we can design a filter to have zeros at particular frequencies, $\angle b$, AND to have poles at particular frequencies, $\angle a$,
- The magnitude $|H(z)|$ is $|z - b|/|z - a|$.
- Using this trick, we can design filters that have much more subtle frequency responses than just an ideal LPF, BPF, or HPF.

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Summary: Poles and Zeros

A first-order autoregressive filter,

$$y[n] = x[n] - bx[n-1] + ay[n-1],$$

has this impulse response, transfer function, and frequency response:

$$h[n] = a^n u[n] - ba^{n-1} u[n-1]$$

$$H(z) = \frac{1 - bz^{-1}}{1 - az^{-1}}$$

$$H(\omega) = \frac{1 - be^{-j\omega}}{1 - ae^{-j\omega}}$$

where a is called the **pole** of the filter, and b is called its **zero**.

Poles and Zeros of the Transfer Function

- The poles and zeros of the transfer function are usually complex numbers, p_k and z_k.
- The frequency response, $H(\omega) = H(z)|_{z=e^{j\omega}}$, has a dip whenever ω equals the phase of any of the zeros, $\omega = \angle z_k$.
- The frequency response, $H(\omega) = H(z)|_{z=e^{j\omega}}$, has a peak whenever ω equals the phase of any of the poles, $\omega = \angle p_k$.