Lecture 28: Z Transform

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ECE 401: Signal and Image Analysis

1. Review: Frequency Response
2. Difference Equations
3. The Z Transform and the Transfer Function
4. Autoregressive Difference Equations
5. Impulse Response and Transfer Function of a First-Order Autoregressive Filter
6. Summary

Outline

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Review: Frequency Response

- **Tones in $\rightarrow$ Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)|\cos(\omega n + \angle H(\omega))$$
$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|H(\omega)|\cos(\omega n + \theta + \angle H(\omega))$$

- where the **Frequency Response** is given by

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

Linearity and Time-Shift Properties

The linearity property of the DTFT says that

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega).$$

The time-shift property says that

$$z[n] = x[n - n_0] \leftrightarrow Z(\omega) = e^{-j\omega n_0} X(\omega)$$

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Difference Equation

A **difference equation** is an equation in terms of time-shifted values of multiple signals. For example,

$$y[n] = x[n] - 2x[n-1] + 2x[n-2]$$

By combining the linearity and time-shift properties of the DTFT, we can translate the whole difference equation into the frequency domain as

$$Y(\omega) = X(\omega) - 2e^{-j\omega}X(\omega) + 2e^{-2j\omega}X(\omega)$$

Impulse Response

A difference equation implements a discrete-time filter. Therefore, it has an **impulse response**. You can find the impulse response by just putting in an impulse, $x[n] = \delta[n]$, and seeing how it responds. Whatever value of $y[n]$ that comes out of the filter is the impulse response:

$$h[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-2] = \begin{cases} 1 & n = 0 \\ -2 & n = 1 \\ 2 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

If you wanted to use `np.convolve` to implement this filter, you now know what impulse response to use.

$$\begin{aligned} y[n] &= x[n] - 2x[n-1] + 2x[n-2] \\ &= \sum_{m=0}^{2} h[m]x[n-m] \end{aligned}$$

Frequency Response

The **frequency response** of a filter is the function $H(\omega)$ such that $Y(\omega) = H(\omega)X(\omega)$, or in other words,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

We can get this from the DTFT of the difference equation:

$$Y(\omega) = \left(1 - 2e^{-j\omega} + 2e^{-2j\omega}\right) X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \left(1 - 2e^{-j\omega} + 2e^{-2j\omega}\right)$$

Frequency Response is DTFT of the Impulse Response

The frequency response is the DTFT of the impulse response.

$$h[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-2]$$
$$H(\omega) = 1 - 2e^{-j\omega} + 2e^{-2j\omega}$$

It sounds like an accident, that frequency response is DTFT of the impulse response. But actually, it's because the method for computing frequency response and the method for computing impulse response do the same two steps, in the opposite order (see next slide).

How to compute the frequency response

1. Take the DTFT of every term:

$$Y(\omega) = X(\omega) - 2e^{-j\omega}X(\omega) + 2e^{-2j\omega}X(\omega)$$

2. Divide by $X(\omega)$.

$$H(\omega) = 1 - 2e^{-j\omega} + 2e^{-2j\omega}$$

How to compute the DTFT of the impulse response

1. Replace $x[n]$ by $\delta[n]$:

$$h[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-2]$$

2. Take the DTFT:

$$H(\omega) = 1 - 2e^{-j\omega} + 2e^{-2j\omega}$$

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Z: a Frequency Variable for Time Shifts

If we're going to be working a lot with delays, instead of pure tones, then it helps to change our frequency variable. Until now, we've been working in terms of ω, the frequency of the pure tone:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

A unit delay, $\delta[n-1]$, has the DTFT $e^{-j\omega}$. Just to reduce the notation a little, let's define the basic unit to be $z = e^{j\omega}$, which is the transform of $\delta[n+1]$ (a unit advance). Then we get:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Main Use of Z Transform: Difference Equations

The main purpose of the Z transform, for now, is just so that we have less to write. Instead of transforming

$$y[n] = x[n] - 2x[n-1] + 2x[n-2]$$

to get

$$Y(\omega) = \left(1 - 2e^{-j\omega} + 2e^{-2j\omega}\right) X(\omega)$$

Now we can just write

$$Y(z) = \left(1 - 2z^{-1} + 2z^{-2}\right) X(z)$$

Transfer function $H(z) = z$-transform version of frequency response

- You can find the **frequency response** of a system by dividing $Y(\omega)/X(\omega)$:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 - 2e^{-j\omega} + 2e^{-2j\omega}$$

- By dividing $Y(z)/X(z)$, we get a function called the **transfer function**.

$$H(z) = \frac{Y(z)}{X(z)} = 1 - 2z^{-1} + 2z^{-2}$$

- **frequency response=transfer function evaluated at** $z = e^{j\omega}$.

Main Use of Z Transform: Difference Equations

The longer the difference equation, the more you will appreciate writing z instead of $e^{j\omega}$.

$$\begin{aligned} y[n] = {} & 0.2x[n+3] + 0.3x[n+2] + 0.5x[n+1] \\ & - 0.5x[n-1] - 0.3x[n-2] - 0.2x[n-3] \end{aligned}$$

$$Y(z) = \left(0.2z^3 + 0.3z^2 + 0.5z^1 - 0.5z^{-1} - 0.3z^{-2} - 0.2z^{-3}\right) X(z)$$

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Autoregressive Filter

Today, instead of filters with this form:

$$y[n] = x[n] - x[n-1]$$

. . . we want to start studying filters of this form:

$$y[n] = x[n] - y[n-1]$$

- This is called **autoregressive**, meaning that $y[n]$ depends on past values (regressive) of itself (auto).
- This is also an **infinite impulse response** (IIR) filter, because the impulse response ($h[n]$) is infinitely long.

Autoregressive Difference Equations

An **autoregressive** filter is one in which the output, $y[n]$, depends on past values of itself (**auto**=self, **regress**=go back). For example,

$$y[n] = x[n] + 0.3x[n-1] + 0.8y[n-1]$$

FIR vs. IIR Filters

- A difference equation with only feedforward terms is always a **finite impulse response (FIR)** filter, because $h[n]$ has finite length.
- An autoregressive filter (with feedback terms) is usually **infinite impulse response (IIR)**, because $h[n]$ is infinitely long.

Causal and Anti-Causal Filters

- If the outputs of a filter depend only on **current and past** values of the input, then the filter is said to be **causal**. An example is

$$y[n] = x[n] + 0.3x[n-1] + 0.8y[n-1]$$

- Any system that depends on both past and future is called **non-causal**. An example is:

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$$

- **Non-causal FIR** filters are easy to analyze, but **non-causal IIR** filters sometimes can't even be implemented. Therefore **all IIR filter in this class will be causal**.

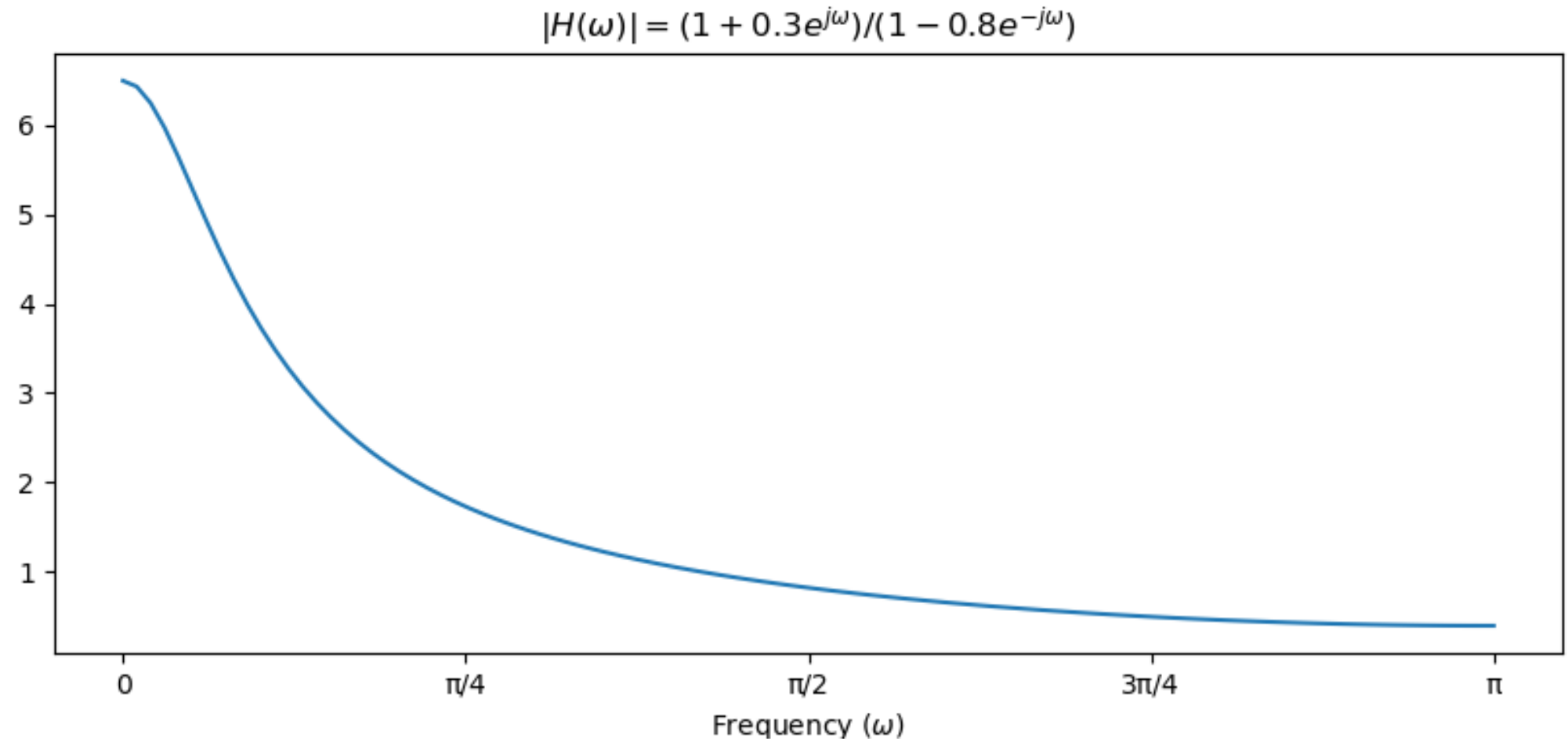
Autoregressive Difference Equations

We can find the frequency response by taking the Z transform or DTFT of each term in the equation:

$$
\begin{aligned}
y[n] &= x[n] + 0.3x[n-1] + 0.8y[n-1] \\
Y(z) &= X(z) + 0.3z^{-1}X(z) + 0.8z^{-1}Y(z) \\
\left(1 - 0.8z^{-1}\right) Y(z) &= \left(1 + 0.3z^{-1}\right) X(z) \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{1 + 0.3z^{-1}}{1 - 0.8z^{-1}} \\
H(\omega) &= H(z)|_{z=e^{j\omega}} = \frac{1 + 0.3e^{-j\omega}}{1 - 0.8e^{-j\omega}}
\end{aligned}
$$

Frequency Response

Here is the frequency response of this filter, plotted using `np.abs((1+0.3*np.exp(-1j*omega))/(1-0.8*np.exp(-1j*omega)))`



General form of an FIR filter

Every FIR filter is just a convolution with a finite-length $h[n]$:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k} h[k]x[n-k]$$

$$h[k] = \begin{cases} b_k & 0 \leq k \leq M \\ 0 & \text{otherwise} \end{cases}$$

This filter has an impulse response ($h[n]$) that is $M+1$ samples long.

- The b_k's are called **feedforward** coefficients, because they feed $x[n]$ forward into $y[n]$.

General form of an IIR filter

For an IIR filter, $h[n]$ has infinite length, so writing out the convolution is inconvenient. Instead, we usually write the **difference equation:**

$$\sum_{\ell=0}^{N} a_\ell y[n-\ell] = \sum_{k=0}^{M} b_k x[n-k]$$

- The a_ℓ's are called **feedback** coefficients, because they feed $y[n]$ back into itself.
- Can we find $h[n]$ in terms of a_ℓ and b_k?

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First-Order Feedback-Only Filter

Let's find the general form of $h[n]$, for the simplest possible autoregressive filter: a filter with one feedback term, and no feedforward terms, like this:

$$y[n] = x[n] + ay[n-1],$$

where a is any constant (positive, negative, real, or complex).

Impulse Response of a First-Order Filter

We can find the impulse response by putting in $x[n] = \delta[n]$, and getting out $y[n] = h[n]$:

$$h[n] = \delta[n] + ah[n-1].$$

Recursive computation gives

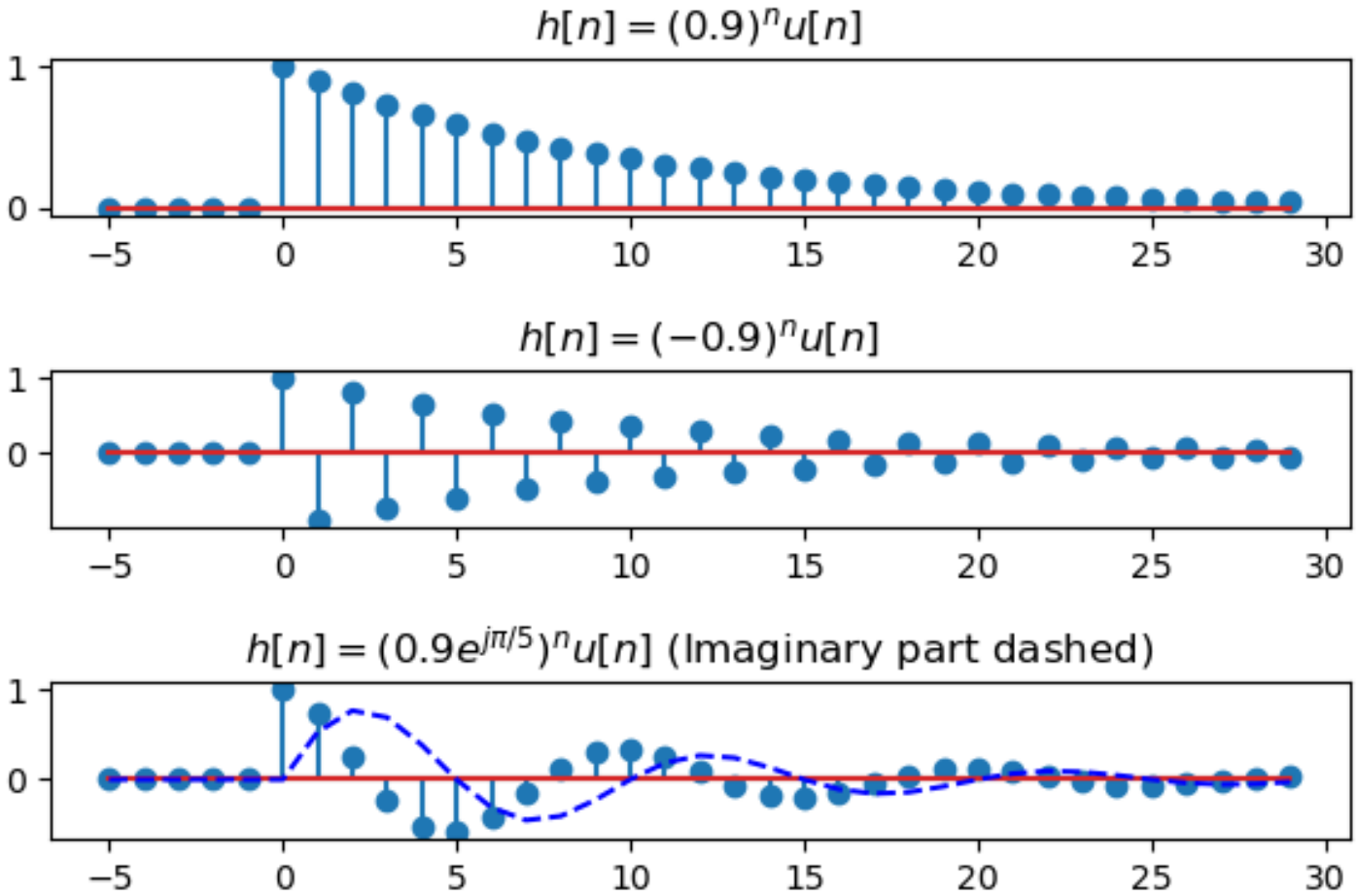
$$\begin{aligned} h[0] &= 1 \\ h[1] &= a \\ h[2] &= a^2 \\ &\vdots \\ h[n] &= a^n u[n] \end{aligned}$$

where we use the notation $u[n]$ to mean the "unit step function,"

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

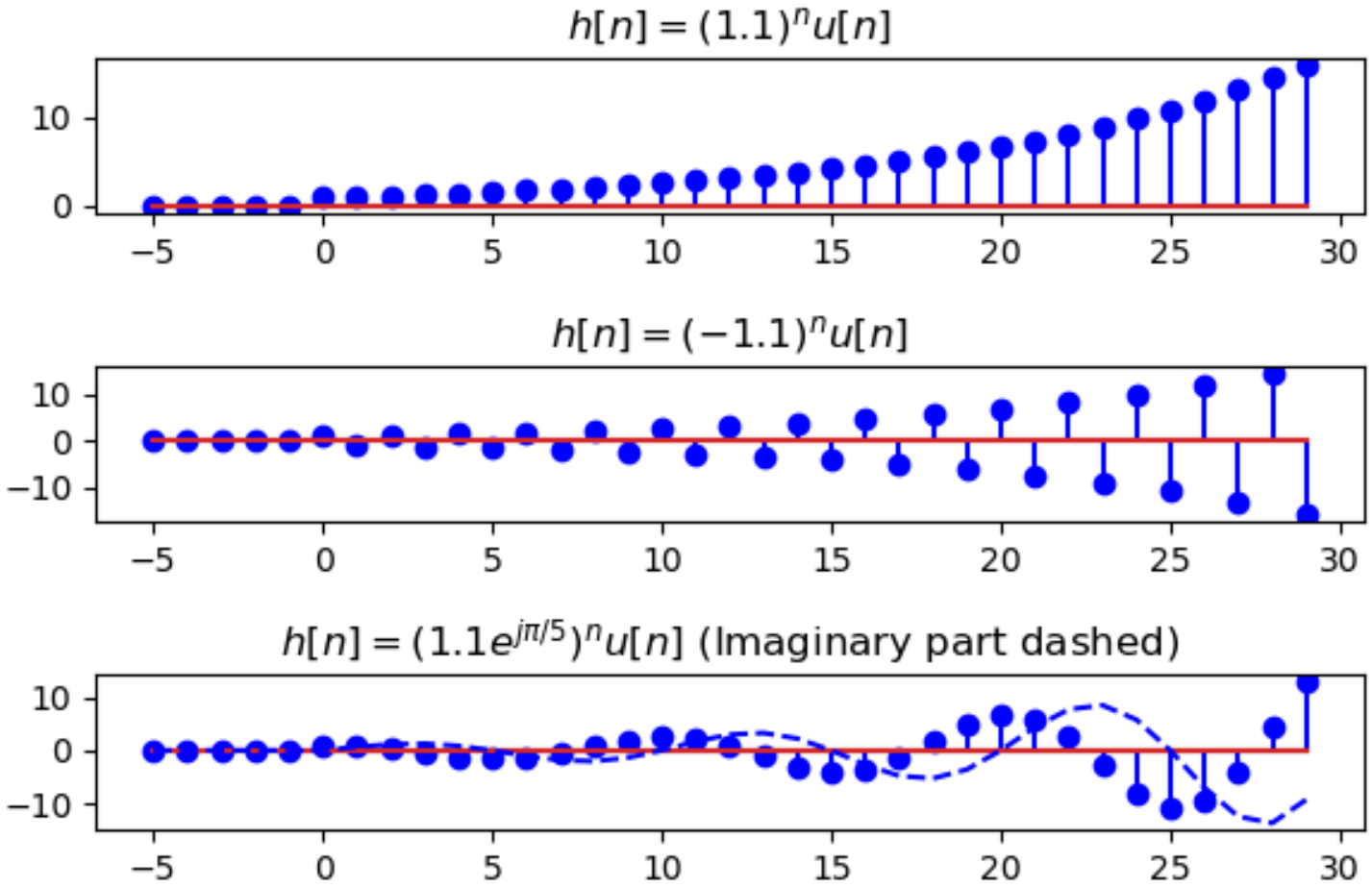
Impulse Response of Stable First-Order Filters

The coefficient, a, can be positive, negative, or even complex. If a is complex, then $h[n]$ is also complex-valued.



Impulse Response of Unstable First-Order Filters

If $|a| > 1$, then the impulse response grows exponentially. If $|a| = 1$, then the impulse response never dies away. In either case, we say the filter is "unstable."



Instability

- A **stable** filter is one that always generates finite outputs ($|y[n]|$ finite) for every possible finite input ($|x[n]|$ finite).
- An **unstable** filter is one that, at least sometimes, generates infinite outputs, even if the input is finite.
- A first-order IIR filter is stable if and only if $|a| < 1$.

Frequency Response of a First-Order Filter

If the filter is stable ($|a| < 1$), then we can find the frequency response by taking the Z transform of $h[n]$:

$$h[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \\ &= \begin{cases} \frac{1}{1-az^{-1}} & \left|az^{-1}\right| < 1 \\ \infty & \left|az^{-1}\right| \geq 1 \end{cases} \end{aligned}$$

$$H(\omega) = \begin{cases} \frac{1}{1-ae^{-j\omega}} & |a| < 1 \\ \infty & |a| \geq 1 \end{cases}$$

Frequency Response of a First-Order Filter

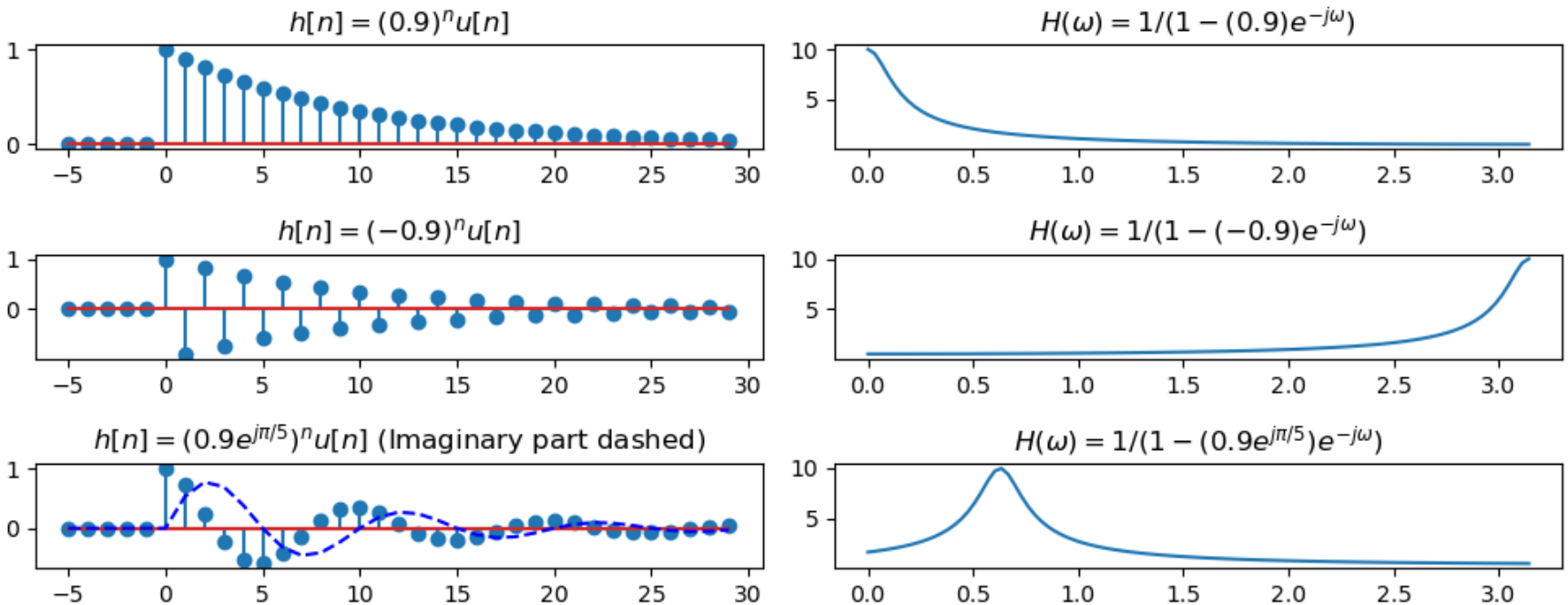
If the filter is stable ($|a| < 1$), then we can also find the frequency response by taking the Z transform of each term in the filter equation:

$$y[n] = x[n] + ay[n-1],$$
$$Y(z) = X(z) + az^{-1}Y(z),$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 - ae^{-j\omega}}$$

That **looks** like it works even if $|a| \geq 1$, but it's a lie. If $|a| \geq 1$, then when you put a pure tone as input, you might get $y[n] = \infty$ as output instead of $y[n] = H(\omega)x[n]$.

Frequency Response of a First-Order Filter

$$H(\omega) = \frac{1}{1 - ae^{-j\omega}} \quad \text{iff } |a| < 1$$



Try the quiz!

Try the quiz!

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Summary: Z Transform

- A **difference equation** is an equation in terms of time-shifted copies of $x[n]$ and/or $y[n]$.
- We have less to write if we use a new frequency variable, $z = e^{j\omega}$. This leads us to the **Z transform**:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- We can find the **transfer function** $H(z) = Y(z)/X(z)$ by taking the Z transform of each term of the difference equation. This will result in the ratio of two polynomials in z.
- We can find the **frequency response** by setting $z = e^{j\omega}$.

Summary: Autoregressive Filter

- An **autoregressive filter** is a filter whose current output, $y[n]$, depends on past values of the output.
- An autoregressive filter is usually **infinite impulse response (IIR)**, because $h[n]$ has infinite length.
- A filter with only feedforward coefficients, and no feedback coefficients, is called **finite impulse response (FIR)**, because $h[n]$ has finite length (its length is just the number of feedforward terms in the difference equation).
- The first-order, feedback-only autoregressive filter has this impulse response and frequency response:

$$h[n] = a^n u[n] \leftrightarrow H(\omega) = \frac{1}{1 - ae^{-j\omega}}$$