

Lecture 20: Windows

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ECE 401: Signal and Image Analysis

- 1 Motivation: Finite Impulse Response (FIR) Filters
- 2 Rectangular Windows
- 3 Bartlett Windows
- 4 Hann and Hamming Windows
- 5 Summary

Outline

- 1 Motivation: Finite Impulse Response (FIR) Filters
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How to create a realizable digital filter

- $L = \text{Odd Length}$:

$$h[n] = h_i[n]w[n],$$

where $w[n]$ is nonzero for $-\left(\frac{L-1}{2}\right) \leq n \leq \left(\frac{L-1}{2}\right)$

- $L = \text{Even Length}$:

$$h[n] = h_i \left[n - \left(\frac{L-1}{2} \right) \right] w[n]$$

where $w[n]$ is nonzero for $0 \leq n \leq L-1$.

Multiplication $\leftrightarrow$ Convolution!

- Convolution $\leftrightarrow$ Multiplication:

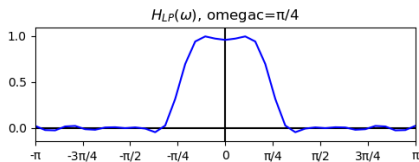
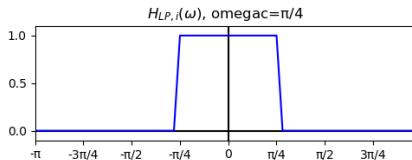
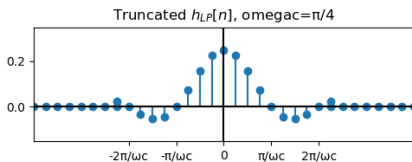
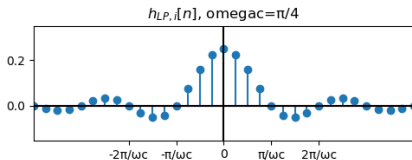
$$h[n] * x[n] \leftrightarrow H(\omega)X(\omega)$$

- Multiplication $\leftrightarrow$ Convolution:

$$w[n]h[n] \leftrightarrow \frac{1}{2\pi} W(\omega) * H(\omega)$$

Result: Windowing Causes Artifacts

We've already seen the result. Windowing by a rectangular window (i.e., truncation) causes nasty artifacts!



Windowing Causes Artifacts

$$h[n] = h_i[n]w[n] \leftrightarrow H(\omega) = \frac{1}{2\pi} H_i(\omega) * W(\omega)$$

Today's Topic:

What is $W(\omega)$? How does it affect $H(\omega)$?

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Review: Rectangle $\leftrightarrow$ Sinc

- The DTFT of a sinc is a rectangle:

$$h[n] = \left(\frac{\omega_c}{\pi}\right) \text{sinc}(\omega_c n) \quad \leftrightarrow \quad H(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

- The DTFT of a rectangle is a sinc-like function, called the Dirichlet form:

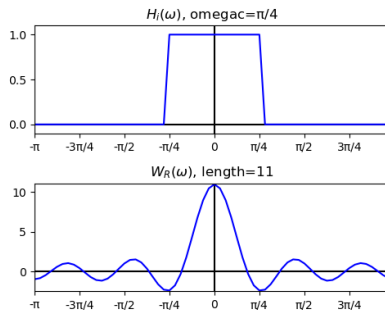
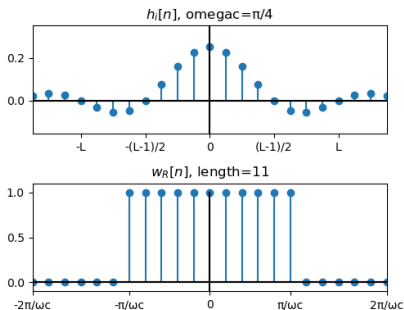
$$w_R[n] = \begin{cases} 1 & |n| \leq \frac{L-1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad W_R(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

Dirichlet Form: Proof Review

Review of the proof:

$$\begin{aligned}W_R(\omega) &= \sum_{n=-\infty}^{\infty} w_R[n] e^{-j\omega n} = \sum_{n=-\frac{L-1}{2}}^{\frac{L-1}{2}} e^{-j\omega n} \\&= e^{j\omega(\frac{L-1}{2})} \sum_{m=0}^{L-1} e^{-j\omega m} \\&= e^{j\omega(\frac{L-1}{2})} \left(\frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} \right) \\&= \left(\frac{e^{j\omega L/2} - e^{-j\omega L/2}}{e^{j\omega/2} - e^{-j\omega/2}} \right) \\&= \left(\frac{\sin(\omega L/2)}{\sin(\omega/2)} \right)\end{aligned}$$

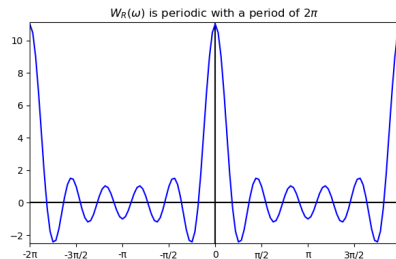
Review: Rectangle $\leftrightarrow$ Sinc



Properties of the Dirichlet form: Periodicity

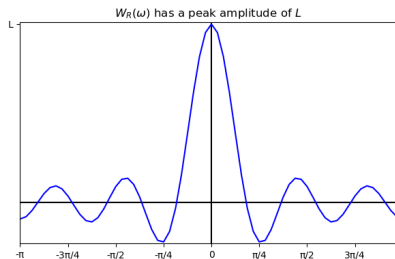
$$W_R(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

Both numerator and denominator are periodic with period 2π .



Properties of the Dirichlet form: DC Value

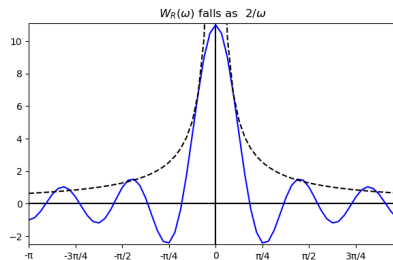
$$W_R(0) = \sum_{n=-\infty}^{\infty} w[n] = L$$



Properties of the Dirichlet form: Sinc-like

$$W_R(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$
$$\approx \frac{\sin(\omega L/2)}{\omega/2}$$

Because, for small values of ω ,
 $\sin\left(\frac{\omega}{2}\right) \approx \frac{\omega}{2}$.



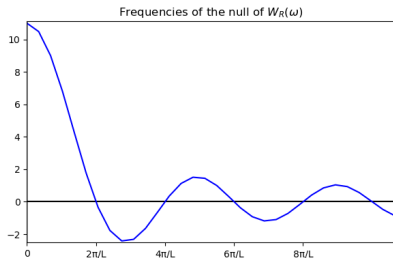
Properties of the Dirichlet form: Nulls

$$W_R(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

It equals zero whenever

$$\frac{\omega L}{2} = k\pi$$

For any nonzero integer, k .



Implication for filter design: Transition band

When $H(\omega) = \frac{1}{2\pi} W(\omega) * H_i(\omega)$, the mainlobe of $W(\omega)$ smooths out the transition from $H(\omega) = 1$ (the “passband”) to $H(\omega) = 0$ (the “stopband”). There is a smooth transition between these two, a kind of ramp, whose width is roughly half the width of $W(\omega)$'s mainlobe, i.e., if

$$H_i(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

then

$$H(\omega) \approx \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c + \frac{2\pi}{L} \end{cases}$$

Properties of the Dirichlet form: Sidelobes

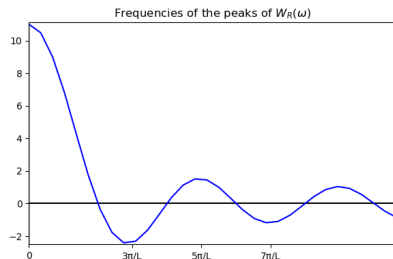
Its sidelobes are

$$W_R\left(\frac{3\pi}{L}\right) = \frac{-1}{\sin(3\pi/2L)} \approx \frac{-2L}{3\pi}$$

$$W_R\left(\frac{5\pi}{L}\right) = \frac{1}{\sin(5\pi/2L)} \approx \frac{2L}{5\pi}$$

$$W_R\left(\frac{7\pi}{L}\right) = \frac{-1}{\sin(7\pi/2L)} \approx \frac{-2L}{7\pi}$$

⋮



Properties of the Dirichlet form: Relative Sidelobe Amplitudes

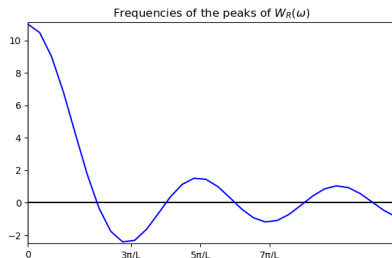
The **relative** sidelobe amplitudes don't depend on L :

$$\frac{W_R\left(\frac{3\pi}{L}\right)}{W_R(0)} = \frac{-1}{L \sin(3\pi/2L)} \approx \frac{-2}{3\pi}$$

$$\frac{W_R\left(\frac{5\pi}{L}\right)}{W_R(0)} = \frac{1}{L \sin(5\pi/2L)} \approx \frac{2}{5\pi}$$

$$\frac{W_R\left(\frac{7\pi}{L}\right)}{W_R(0)} = \frac{-1}{L \sin(7\pi/2L)} \approx \frac{-2}{7\pi}$$

⋮



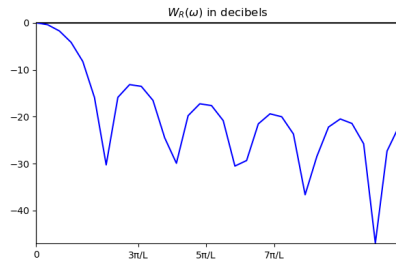
Properties of the Dirichlet form: Decibels

We often describe the relative sidelobe amplitudes in decibels, which are defined as

$$20 \log_{10} \left| \frac{W\left(\frac{3\pi}{L}\right)}{W(0)} \right| \approx 20 \log_{10} \frac{2}{3\pi} \approx -13\text{dB}$$

$$20 \log_{10} \left| \frac{W\left(\frac{5\pi}{L}\right)}{W(0)} \right| \approx 20 \log_{10} \frac{2}{5\pi} \approx -18\text{dB}$$

$$20 \log_{10} \left| \frac{W\left(\frac{7\pi}{L}\right)}{W(0)} \right| \approx 20 \log_{10} \frac{2}{7\pi} \approx -21\text{dB}$$

$$\vdots$$


Implication for filter design: Ripple

- The d^{th} sidelobe of $W(\omega)$ has an amplitude of $2L/(2k+1)\pi$, and a width of $2\pi/L$, so its total area is roughly $4/(2k+1)$ — regardless of the length of the window!
- As ω moves away from the transition band, the number of sidelobes of $W(\omega)$ overlapping with the passband of $H_i(\omega)$ decreases, so the filter response $H(\omega)$ ripples positive and negative.
- **Stopband ripples** are frequencies where $H_i(\omega) = 0$, but $H(\omega) \neq 0$ because of the ripple.
- Longer windows result in filters with smaller ripples.

Quiz

Go to the course webpage, and try the quiz!

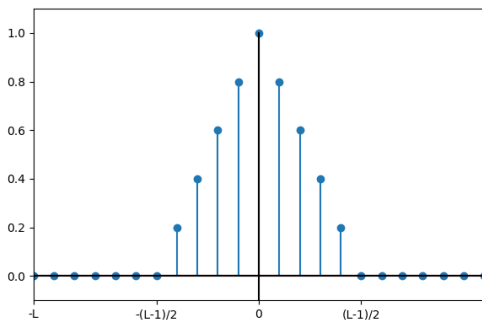
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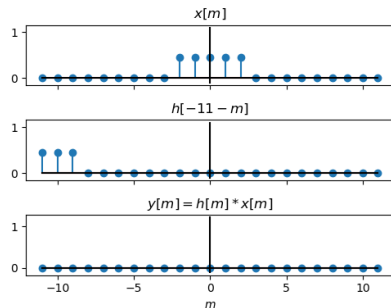
Bartlett (Triangular) Window

A Bartlett window is a triangle:

$$w_B[n] = \max\left(0, 1 - \frac{|n|}{(L-1)/2}\right)$$



A Bartlett window is the convolution of two rectangular windows, each with a height of $\sqrt{\frac{2}{L-1}}$ and a length of $\frac{L-1}{2}$.

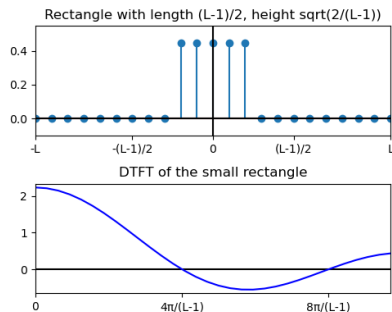


Since each of the two little rectangles has a height of $\sqrt{\frac{2}{L-1}}$ and a length of $\frac{L-1}{2}$, their spectra have a DC value of

$$W_B(0) = \sqrt{\frac{L-1}{2}},$$

and nulls of

$$W_B\left(\frac{4\pi k}{L-1}\right) = 0$$

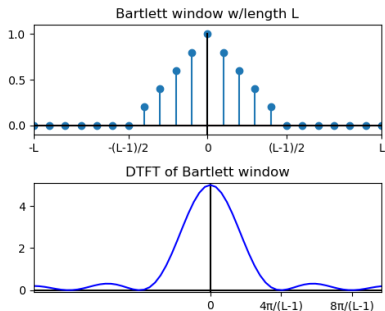


Since

$$w_B[n] = w_R[n] * w_R[n],$$

therefore

$$W_B(\omega) = (W_R(\omega))^2$$

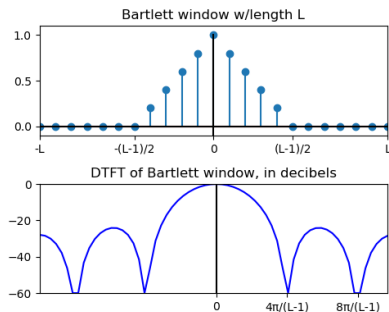


In particular: the sidelobes of a Bartlett window are much lower than those of a rectangular window!

$$20 \log_{10} \left| \frac{W_B \left(\frac{6\pi}{L-1} \right)}{W(0)} \right| \approx -26\text{dB}$$

$$20 \log_{10} \left| \frac{W \left(\frac{10\pi}{L-1} \right)}{W(0)} \right| \approx -36\text{dB}$$

⋮



Things to Notice

- The **main lobe width** has been doubled, because the Bartlett window is created by convolving two half-length rectangular windows.
 - Therefore $H(\omega) = \frac{1}{2\pi} W_N(\omega) * H_i(\omega)$ will have a wider transition band.
- The **sidelobe height** has been dramatically reduced, because convolving in the time domain means multiplying in the frequency domain.
 - Therefore $H(\omega) = \frac{1}{2\pi} W_N(\omega) * H_i(\omega)$ will have much lower stopband ripple.

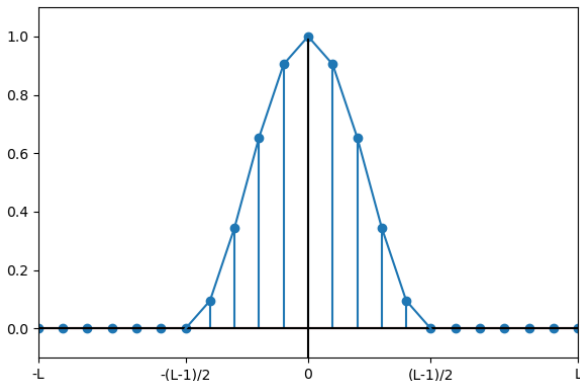
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The Hann Window

Here's the Hann window:

$$w_N[n] = w_R[n] \left(\frac{1}{2} + \frac{1}{2} \cos \left(\frac{2\pi n}{L-1} \right) \right)$$



Spectrum of the Hann Window

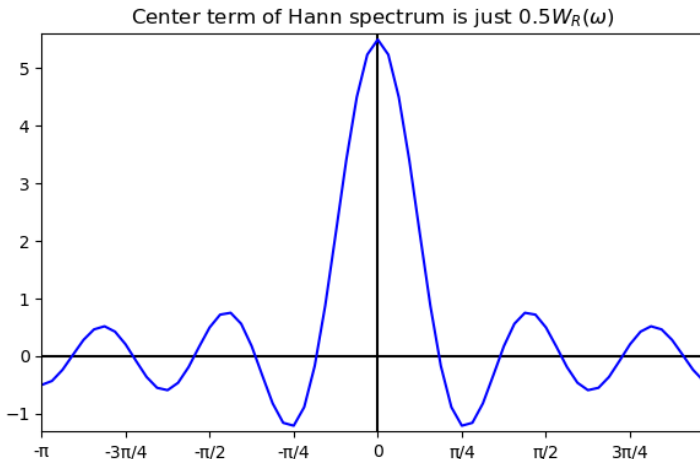
$$\begin{aligned}w_N[n] &= w_R[n] \left(\frac{1}{2} + \frac{1}{2} \cos \left(\frac{2\pi n}{L-1} \right) \right) \\&= \frac{1}{2} w_R[n] + \frac{1}{4} w_R[n] e^{-j \frac{2\pi n}{L-1}} + \frac{1}{4} w_R[n] e^{+j \frac{2\pi n}{L-1}}\end{aligned}$$

So its spectrum is:

$$W_N(\omega) = \frac{1}{2} W_R(\omega) + \frac{1}{4} W_R \left(\omega - \frac{2\pi}{L-1} \right) + \frac{1}{4} W_R \left(\omega + \frac{2\pi}{L-1} \right)$$

Spectrum of the Rectangular Window

Here's the DTFT of the rectangular window, $0.5W_R(\omega)$:

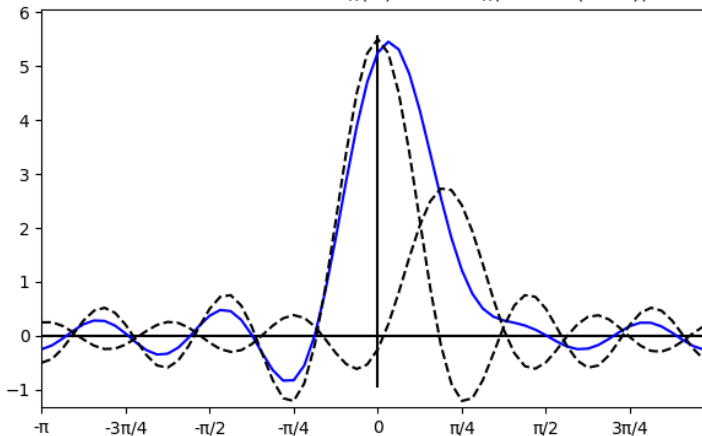


Spectrum of Two Parts of the Hann Window

Here's the DTFT of two parts of the Hann Window,

$$\frac{1}{2}W_R(\omega) + \frac{1}{4}W_R\left(\omega - \frac{2\pi}{L-1}\right):$$

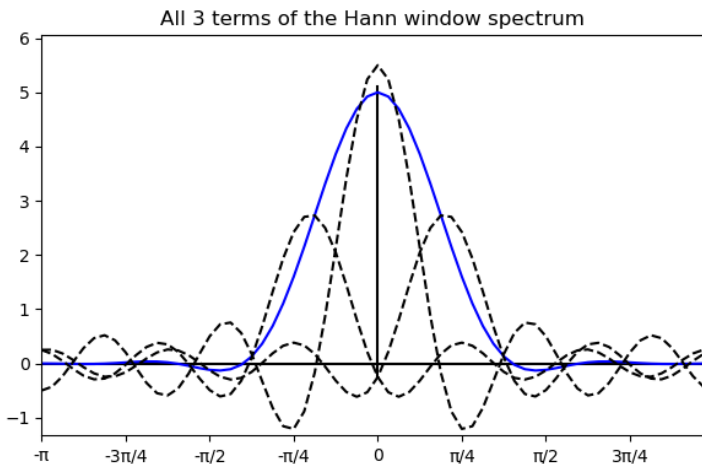
First two terms are $0.5W_R(\omega) + 0.25W_R(\omega - 2\pi/(L-1))$



Spectrum of the Hann Window

Here's the DTFT of the Hann window,

$$W_N(\omega) = \frac{1}{2} W_R(\omega) + \frac{1}{4} W_R\left(\omega - \frac{2\pi}{L-1}\right) + \frac{1}{4} W_R\left(\omega + \frac{2\pi}{L-1}\right):$$



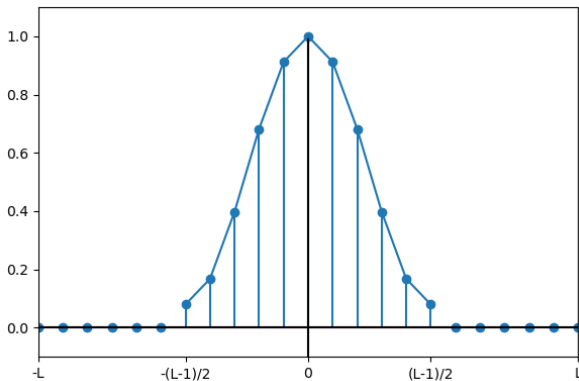
Things to Notice

- The **main lobe width** has been doubled, because each of the two nulls next to the main lobe have been canceled out.
 - Therefore $H(\omega) = \frac{1}{2\pi} W_N(\omega) * H_i(\omega)$ will have a wider transition band.
- The **sidelobe height** has been dramatically reduced, because the frequency-shifted copies each cancel out the main copy.
 - Therefore $H(\omega) = \frac{1}{2\pi} W_N(\omega) * H_i(\omega)$ will have much lower stopband ripple.

The Hamming Window

Here's the Hamming window:

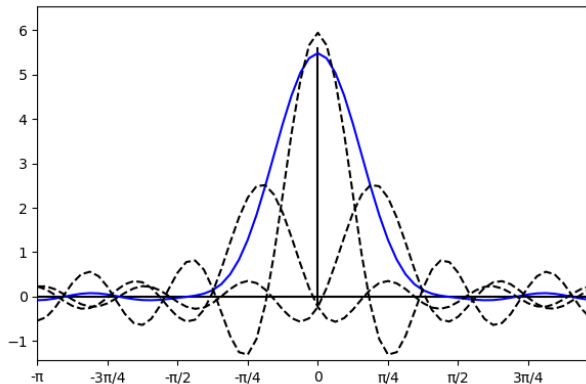
$$w_M[n] = w_R[n] \left(A + (1 - A) \cos \left(\frac{2\pi n}{L - 1} \right) \right)$$



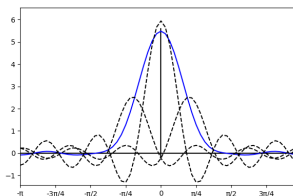
Spectrum of the Hamming Window

$$W_M(\omega) = AW_R(\omega) + \frac{1-A}{2}W_R\left(\omega - \frac{2\pi}{L-1}\right) + \frac{1-A}{2}W_R\left(\omega + \frac{2\pi}{L-1}\right),$$

where A is chosen to minimize the height of the first sidelobe:



Spectrum of the Hamming Window



The first sidelobe is at $\omega = \frac{5\pi}{L}$. At that frequency, $W_M(\omega)$ is roughly:

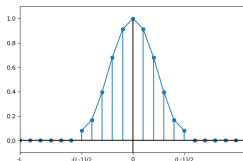
$$\begin{aligned}
 & AW_R\left(\frac{5\pi}{L}\right) + \frac{1-A}{2}W_R\left(\frac{5\pi}{L} - \frac{2\pi}{L}\right) + \frac{1-A}{2}W_R\left(\frac{5\pi}{L} + \frac{2\pi}{L}\right) \\
 & \approx A\left(\frac{L}{5\pi}\right) - \frac{1-A}{2}\left(\frac{L}{3\pi}\right) - \frac{1-A}{2}\left(\frac{L}{7\pi}\right) \\
 & \approx (0.13945A - 0.07579)L,
 \end{aligned}$$

... which is zero if $A = 0.5434782$.

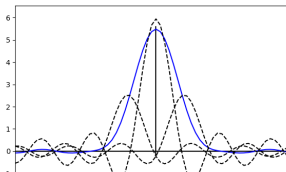
The Hamming Window

The Hamming window chooses $A = 0.5434782$, rounded off to two significant figures:

$$w_M[n] = w_R[n] \left(0.54 + 0.46 \cos \left(\frac{2\pi n}{L-1} \right) \right)$$



... with the result that the first sidelobe of the Hamming window has an amplitude below 0.01:



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Main Features of Four Windows

Window	Shape	First Null ($\approx$ Transition Band-width)	First Side-lobe ($\approx$ Stopband Ripple)	First Side-lobe Level
Rectangular	rectangle	$\frac{2\pi}{L}$	$0.11L$	-13dB
Bartlett	triangle	$\frac{4\pi}{L}$	$0.05L$	-26dB
Hann	raised cosine	$\frac{4\pi}{L}$	$-0.028L$	-31dB
Hamming	raised cosine	$\frac{4\pi}{L}$	$-0.0071L$	-43dB