

Lecture 16: Discrete-Time Fourier Transform (DTFT)

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ECE 401: Signal and Image Analysis

- 1 Review: Frequency Response
- 2 Discrete Time Fourier Transform
- 3 Properties of the DTFT
- 4 The Local Averaging Filters
- 5 Rectangular Windows
- 6 Summary

Outline

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Response of LSI System to Periodic Inputs

Suppose we compute $y[n] = x[n] * h[n]$, where

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}, \text{ and}$$
$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] e^{j2\pi kn/N}.$$

The relationship between $Y[k]$ and $X[k]$ is given by the frequency response:

$$Y[k] = H(k\omega_0)X[k]$$

where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

Response of LSI System to Aperiodic Inputs

But what about signals that never repeat themselves?

Can we still write something like

$$Y(\omega) = H(\omega)X(\omega)?$$

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Aperiodic

An “aperiodic signal” is a signal that is not periodic.

- Music: strings, woodwinds, and brass are periodic, drums and rain sticks are aperiodic.
- Speech: vowels and nasals are periodic, plosives and fricatives are aperiodic.
- Images: stripes are periodic, clouds are aperiodic.
- Bioelectricity: heartbeat is periodic, muscle contractions are aperiodic.

Periodic

The spectrum of a periodic signal is given by its Fourier series. In discrete time, that's:

$$\begin{aligned} X_k &= \frac{1}{N_0} \sum_{n=-\frac{N_0}{2}}^{\frac{N_0-1}{2}} x[n] e^{-j\frac{2\pi kn}{N_0}} \\ &= \frac{1}{N_0} \sum_{n=-\frac{N_0}{2}}^{\frac{N_0-1}{2}} x[n] e^{-j\omega n} \end{aligned}$$

and that gives the frequency content of the signal, at the frequency $\omega = \frac{2\pi k}{N_0}$.

Here I'm using $n \in \left\{-\frac{N_0}{2}, \dots, \frac{N_0-1}{2}\right\}$, but the sum could be over any sequence of N_0 continuous samples.

Aperiodic

An aperiodic signal is one that **never** repeats itself. So we want something like the limit, as $N_0 \rightarrow \infty$, of the Fourier series. Here is the simplest such thing that is useful:

Discrete-Time Fourier Transform (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Fourier Series vs. Fourier Transform

The Fourier Series coefficients are:

$$X_k = \frac{1}{N_0} \sum_{n=-\frac{N_0}{2}}^{\frac{N_0-1}{2}} x[n] e^{-j\omega n}$$

The Fourier transform is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Notice that, besides taking the limit as $N_0 \rightarrow \infty$, we also got rid of the $\frac{1}{N_0}$ factor. So we can think of the DTFT as

$$X(\omega) = \lim_{N_0 \rightarrow \infty, \omega = \frac{2\pi k}{N_0}} N_0 X_k$$

where the limit is: as $N_0 \rightarrow \infty$, and $k \rightarrow \infty$, but $\omega = \frac{2\pi k}{N_0}$ remains constant.

Inverse DTFT

In order to convert $X(\omega)$ back to $x[n]$, we'll take advantage of orthogonality:

$$\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi & m = n \\ 0 & (m - n) = \text{any nonzero integer} \end{cases}$$

Inverse DTFT

Taking advantage of orthogonality, we can see that

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right) e^{j\omega m} d\omega \\&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} x[n] \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega \\&= x[m]\end{aligned}$$

Fourier Series and Fourier Transform

Discrete-Time Fourier Series (DTFS):

$$X_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j \frac{2\pi kn}{N_0}}$$

$$x[n] = \sum_{k=0}^{N_0-1} X_k e^{j \frac{2\pi kn}{N_0}}$$

Discrete-Time Fourier Transform (DTFT):

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Quiz

Go to the course webpage, try the quiz!

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Properties of the DTFT

In order to better understand the DTFT, let's discuss these properties:

- ① Periodicity
- ② Linearity
- ③ Time Shift
- ④ Frequency Shift
- ⑤ Filtering is Convolution

Property #4 is actually the reason why we invented the DTFT in the first place. Before we discuss it, though, let's talk about the others.

0. Periodicity

The DTFT is periodic with a period of 2π . That's just because $e^{j2\pi} = 1$:

$$X(\omega) = \sum_n x[n] e^{-j\omega n}$$

$$X(\omega + 2\pi) = \sum_n x[n] e^{-j(\omega+2\pi)n} = \sum_n x[n] e^{-j\omega n} = X(\omega)$$

$$X(\omega - 2\pi) = \sum_n x[n] e^{-j(\omega-2\pi)n} = \sum_n x[n] e^{-j\omega n} = X(\omega)$$

For example, the inverse DTFT can be defined in two different ways:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega$$

Those two integrals are equal because $X(\omega + 2\pi) = X(\omega)$.

1. Linearity

The DTFT is linear:

$$z[n] = ax[n] + by[n] \quad \leftrightarrow \quad Z(\omega) = aX(\omega) + bY(\omega)$$

Proof:

$$\begin{aligned} Z(\omega) &= \sum_n z[n] e^{-j\omega n} \\ &= a \sum_n x[n] e^{-j\omega n} + b \sum_n y[n] e^{-j\omega n} \\ &= aX(\omega) + bY(\omega) \end{aligned}$$

2. Time Shift Property

Shifting in time is the same as multiplying by a complex exponential in frequency:

$$z[n] = x[n - n_0] \quad \leftrightarrow \quad Z(\omega) = e^{-j\omega n_0} X(\omega)$$

Proof:

$$\begin{aligned} Z(\omega) &= \sum_{n=-\infty}^{\infty} x[n - n_0] e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+n_0)} \quad (\text{where } m = n - n_0) \\ &= e^{-j\omega n_0} X(\omega) \end{aligned}$$

3. Frequency Shift Property

Shifting in frequency is the same as multiplying by a complex exponential in time:

$$z[n] = x[n]e^{j\omega_0 n} \quad \leftrightarrow \quad Z(\omega) = X(\omega - \omega_0)$$

Proof:

$$\begin{aligned} Z(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega_0 n}e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n} \\ &= X(\omega - \omega_0) \end{aligned}$$

4. Convolution Property

Convolving in time is the same as multiplying in frequency:

$$y[n] = h[n] * x[n] \quad \leftrightarrow \quad Y(\omega) = H(\omega)X(\omega)$$

Proof: Remember that $y[n] = h[n] * x[n]$ means that $y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$. Therefore,

$$\begin{aligned} Y(\omega) &= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} h[m]x[n-m] \right) e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (h[m]x[n-m]) e^{-j\omega m} e^{-j\omega(n-m)} \\ &= \left(\sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} \right) \left(\sum_{(n-m)=-\infty}^{\infty} x[n-m]e^{-j\omega(n-m)} \right) \\ &= H(\omega)X(\omega) \end{aligned}$$

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Local Average Filters

Let's go back to the local averaging filter. I want to define two different types of local average: centered, and delayed.

- **Centered local average:** This one averages $\left(\frac{L-1}{2}\right)$ future samples, $\left(\frac{L-1}{2}\right)$ past samples, and $x[n]$:

$$y_c[n] = \frac{1}{L} \sum_{m=-\left(\frac{L-1}{2}\right)}^{\left(\frac{L-1}{2}\right)} x[n-m]$$

- **Causal local average:** This one averages $x[n]$ and $L-1$ of its past samples:

$$y_d[n] = \frac{1}{L} \sum_{m=0}^{L-1} x[n-m]$$

Notice that $y_d[n] = y_c\left[n - \left(\frac{L-1}{2}\right)\right]$.

Local Average Filters

We can write both of these as filters:

- **Centered local average:**

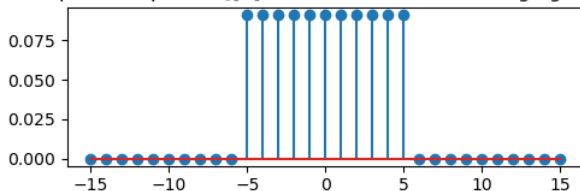
$$y_c[n] = f_c[n] * x[n]$$
$$f_c[n] = \begin{cases} \frac{1}{L} & -\left(\frac{L-1}{2}\right) \leq n \leq \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

- **Causal local average:**

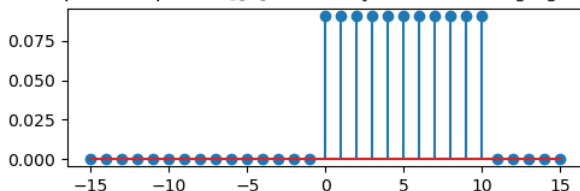
$$y_d[n] = f_d[n] * x[n]$$
$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases}$$

Local Average Filters

Impulse response $f_c[n]$ of a centered local averaging filter



Impulse response $f_d[n]$ of a delayed local averaging filter



Notice that $f_d[n] = f_c \left[n - \left(\frac{L-1}{2} \right) \right]$.

The relationship between centered local average and delayed local average

Notice that $f_d[n] = f_c[n - \frac{L-1}{2}]$. We can find the relationship between their DTFTs using variable substitution, with the variable $m = n - \frac{L-1}{2}$:

$$\begin{aligned} F_d(\omega) &= \sum_{n=-\infty}^{\infty} f_d[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} f_c[n - \frac{L-1}{2}] e^{-j\omega n} \\ &= e^{-j\omega \frac{L-1}{2}} \sum_{m=-\infty}^{\infty} f_c[m] e^{-j\omega m} \\ &= e^{-j\omega \frac{L-1}{2}} F_c(\omega) \end{aligned}$$

The frequency response of a local average filter

Let's find the frequency response of

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \leq m \leq L-1 \\ 0 & \text{otherwise} \end{cases}$$

The formula is

$$F_d(\omega) = \sum_m f[m] e^{-j\omega m},$$

so,

$$F_d(\omega) = \sum_{m=0}^{L-1} \frac{1}{L} e^{-j\omega m}$$

The frequency response of a local average filter

$$F_d(\omega) = \sum_{m=0}^{L-1} \frac{1}{L} e^{-j\omega m}$$

This is just a standard geometric series,

$$\sum_{m=0}^{L-1} a^m = \frac{1 - a^L}{1 - a},$$

so:

$$F_d(\omega) = \frac{1}{L} \left(\frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} \right)$$

The frequency response of a local average filter

We now have an extremely useful transform pair:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \leq m \leq L-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow F_d(\omega) = \frac{1}{L} \left(\frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} \right)$$

Let's attempt to convert that into polar form, so we can find magnitude and phase response. Notice that both the numerator and the denominator are subtractions of complex numbers, so we might be able to use $2j \sin(x) = e^{jx} - e^{-jx}$ for some x . Let's try:

$$\begin{aligned} \frac{1}{L} \left(\frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} \right) &= \frac{1}{L} \frac{e^{-j\omega L/2}}{e^{-j\omega/2}} \left(\frac{e^{j\omega L/2} - e^{-j\omega L/2}}{e^{j\omega/2} - e^{-j\omega/2}} \right) \\ &= e^{-j\omega(\frac{L-1}{2})} \frac{1}{L} \left(\frac{2j \sin(\omega L/2)}{2j \sin(\omega/2)} \right) \\ &= e^{-j\omega(\frac{L-1}{2})} \frac{1}{L} \left(\frac{\sin(\omega L/2)}{\sin(\omega/2)} \right) \end{aligned}$$

The frequency response of a local average filter

Now we have $F_d(\omega)$ in almost magnitude-phase form:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \leq m \leq L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \left(\frac{\sin(\omega L/2)}{L \sin(\omega/2)} \right) e^{-j\omega(\frac{L-1}{2})}$$

By the way, remember we discovered that

$$F_d(\omega) = e^{-j\omega(\frac{L-1}{2})} F_c(\omega)$$

Notice anything?

DTFT of Local Averaging Filters

- Centered local average:

$$f_c[n] = \begin{cases} \frac{1}{L} & -(\frac{L-1}{2}) \leq n \leq (\frac{L-1}{2}) \\ 0 & \text{otherwise} \end{cases}$$

$$F_c(\omega) = \frac{\sin(\omega L/2)}{L \sin(\omega/2)}$$

- Causal local average:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases} \quad F_d(\omega) = \frac{\sin(\omega L/2)}{L \sin(\omega/2)} e^{-j\omega(\frac{L-1}{2})}$$

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DTFT of Rectangular Window

The local summing filter is just a scaled version of the local averaging filter. It is commonly called the “rectangular window,” for reasons that we’ll explore in future lectures.

- **Centered rectangular window:**

$$w_c[n] = \begin{cases} 1 & -(\frac{L-1}{2}) \leq n \leq (\frac{L-1}{2}) \\ 0 & \text{otherwise} \end{cases}$$

$$W_c(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

- **Causal rectangular window:**

$$w_d[n] = \begin{cases} 1 & 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases} \quad W_d(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(\frac{L-1}{2})}$$

Dirichlet form

Of all four of these signals, the centered rectangular window has the simplest DTFT. The textbook calls this signal the “Dirichlet form:”

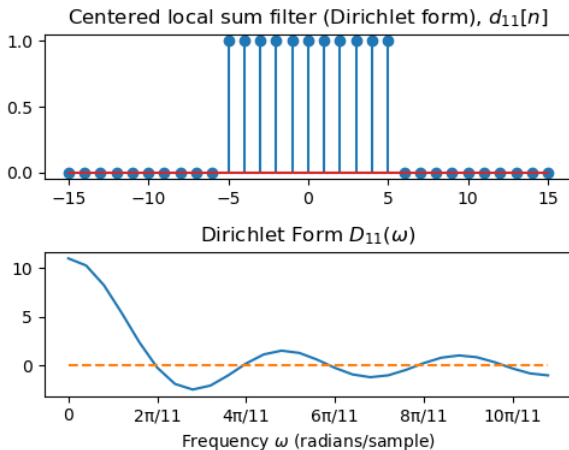
$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

That is, exactly, the frequency response of a centered rectangular window:

$$d_L[n] = w_c[n] = \begin{cases} 1 & -(\frac{L-1}{2}) \leq n \leq (\frac{L-1}{2}) \\ 0 & \text{otherwise} \end{cases}$$

Dirichlet form

Here's what it looks like:



Dirichlet form

Since every local averaging filter is based on Dirichlet form, it's worth spending some time to understand it better.

$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

- It's equal to zero every time $\omega L/2$ is a multiple of π . So

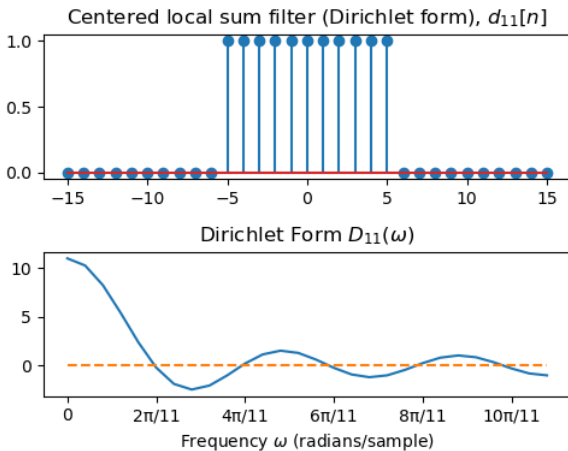
$$D_L\left(\frac{2\pi k}{L}\right) = 0 \text{ for all integers } k \text{ except } k = 0$$

- At $\omega = 0$, the value of $\frac{\sin(\omega L/2)}{\sin(\omega/2)}$ is undefined, but it's possible to prove that $\lim_{\omega \rightarrow 0} D_L(\omega) = L$. To make life easy, we'll just define it that way:

DEFINE: $D_L(0) = L$

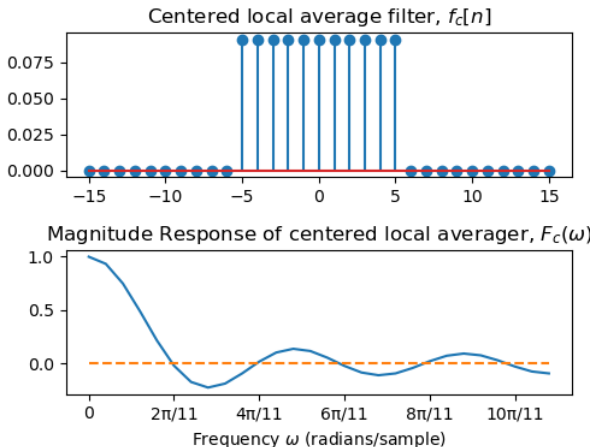
Dirichlet form

Here's what it looks like:



Local averaging filter

Here's what the centered local averaging filter looks like. Notice that it's just $1/L$ times the Dirichlet form:



Outline

- 1 Review: Frequency Response
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Summary

The DTFT (discrete time Fourier transform) of any signal is $X(\omega)$, given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n - n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$

Properties of the DTFT

Properties worth knowing include:

- ① Periodicity: $X(\omega + 2\pi) = X(\omega)$
- ① Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$$

- ② Time Shift: $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
- ③ Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
- ④ Filtering is Convolution:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Summary: DTFTs of Rectangular Windows

- The **Centered Local Averaging Filter** is $1/L$ times the Dirichlet form:

$$f_c[n] = \begin{cases} \frac{1}{L} & -(\frac{L-1}{2}) \leq n \leq (\frac{L-1}{2}) \\ 0 & \text{otherwise} \end{cases} \leftrightarrow F_c(\omega) = \frac{\sin(\omega L/2)}{L \sin(\omega/2)}$$

- The **Causal Local Averaging Filter** is $f_c[n]$, delayed by $\frac{L-1}{2}$ samples:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow F_d(\omega) = \frac{\sin(\omega L/2)}{L \sin(\omega/2)} e^{-j\omega(\frac{L-1}{2})}$$