

Lecture 14: Exam 1 Review

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ECE 401: Signal and Image Analysis

- 1 Exam administration
- 2 Topics included in the exam
- 3 Sample problems

Outline

- 1 Exam administration
- 2 Topics included in the exam
- 3 Sample problems

When, Where

- When: Monday, October 5, 1:00-1:50pm.
- Where: ECEB 3015.
- Online section: Be signed on to zoom 12:55pm, with your camera on. Have Gradescope open on your phone or device, so you can upload your answers promptly at 1:50pm when the exam is completed.

What to bring

- Bring: Pencil or pen. Erasers.
- Notes sheet: You'll be given one formula sheet. You are also permitted to bring your own formula sheet. Yours should be 8.5x11, handwritten, both sides.
- Don't bring: Anything electronic. Any of your own paper (the last page of the exam paper will be blank, so you can use it as scratch paper).

Grading

- Exam will be graded on Gradescope.
- There will probably be four problems, each worth about 25 points.
- Grades should be available one week after the exam.

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Topics included in the exam

- 1 Systems: linearity, time-invariance, convolution, causality, stability
- 2 Sampling: aliasing, interpolation, DT implementation of CT systems
- 3 Fourier series: F0, Fourier series, DFT
- 4 Frequency response, circular convolution, time-domain aliasing

Linearity and time invariance

- A system is **linear** if and only if, for any two inputs $x_1[n]$ and $x_2[n]$ that produce outputs $y_1[n]$ and $y_2[n]$,

$$x[n] = x_1[n] + x_2[n] \xrightarrow{\mathcal{H}} y[n] = y_1[n] + y_2[n]$$

- A system is **shift-invariant** if and only if, for any input $x_1[n]$ that produces output $y_1[n]$,

$$x[n] = x_1[n - n_0] \xrightarrow{\mathcal{H}} y[n] = y_1[n - n_0]$$

Convolution

If a system is **linear and shift-invariant** (LSI), then it can be implemented using convolution:

$$y[n] = h[n] * x[n] = \sum_m h[m]x[n-m] = \sum_m h[n-m]x[m]$$

where $h[n]$ is the impulse response:

$$\delta[n] \xrightarrow{\mathcal{H}} h[n]$$

Causality and Stability

- A causal system is one whose outputs depend only on current and past inputs, not future inputs.
- An LTI system is causal **if and only if** $h[n]$ is right-sided.
- A system is stable if and only if every bounded input produces a bounded output.
- An LTI system is stable **if and only if** $h[n]$ is magnitude-summable.

Sampling and aliasing

- A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, $f < \frac{F_s}{2}$.
- If the Nyquist criterion is violated, then:
 - If $\frac{F_s}{2} < f < F_s$, then it will be aliased to

$$f_a = F_s - f$$

$$z_a = z^*$$

i.e., the sign of all sines will be reversed.

- If $F_s < f < \frac{3F_s}{2}$, then it will be aliased to

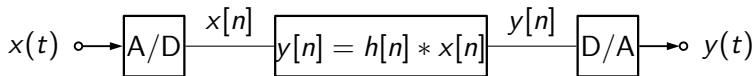
$$f_a = f - F_s$$

$$z_a = z$$

Interpolation

- Piece-wise constant interpolation = interpolate using a rectangle
- Piece-wise linear interpolation = interpolate using a triangle
- Cubic-spline interpolation = interpolate using a spline
- Ideal interpolation = interpolate using a sinc

DT Implementation of CT Systems



If $x(t)$ is periodic, then $y(t)$ is also periodic with the same period but different Fourier Series coefficients:

$$x(t) = \sum_{k=-N}^N X_k e^{j2\pi k F_0 t}, \quad x[n] = \sum_{k=-N}^N X_k e^{jk\omega_0 n}$$
$$y[n] = \sum_{k=-N}^N Y_k e^{jk\omega_0 n}, \quad y(t) = \sum_{k=-N}^N Y_k e^{j2\pi k F_0 t}$$

Pitch

Fourier's Theorem: Any periodic waveform, $x(t + T_0) = x(t)$, can be synthesized as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

where $F_0 = \frac{1}{T_0}$ is the fundamental frequency.

Fourier Series

$$X_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0} dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

- **Periodicity and linearity:**

$$x(t) = x(t + T_0)$$

$$ax(t) + by(t) \leftrightarrow aX_k + bY_k$$

- **Time & Frequency Shift (Subject to periodicity):**

$$x(t) e^{j\frac{2\pi k_0 t}{T_0}} \leftrightarrow X_{k-k_0}$$

$$x(t - \tau) \leftrightarrow X_k e^{-j\frac{2\pi k \tau}{T_0}}$$

Discrete Fourier Transform

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

- **Periodicity and linearity:**

$$x[n] = x[n + N], \quad X[k] = X[k + N]$$

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1[k] + bX_2[k]$$

- **Time & Frequency Shift (Subject to periodicity):**

$$x[n] e^{j\frac{2\pi k_0 n}{N}} \leftrightarrow X[k - k_0]$$

$$x[n - n_0] \leftrightarrow X[k] e^{-j\frac{2\pi kn_0}{N}}$$

Frequency Response

- **Tones in → Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n}$$

$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)| \cos(\omega n + \angle H(\omega))$$

$$x[n] = A \cos(\omega n + \theta) \rightarrow y[n] = A|H(\omega)| \cos(\omega n + \theta + \angle H(\omega))$$

- where the **Frequency Response** is given by

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

Circular convolution

Multiplying the DFTs, instead of implementing convolution directly, reduces computation from $\mathcal{O}\{N^2\}$ to $\mathcal{O}\{N \log N\}$. However, multiplying the DFT means **circular convolution** of the time-domain signals:

$$y[n] = h[n] \circledast x[n] \leftrightarrow Y[k] = H[k]X[k],$$

Circular convolution ($h[n] \circledast x[n]$) is defined like this:

$$h[n] \circledast x[n] = \sum_{m=0}^{N-1} x[m] h[((n-m))_N] = \sum_{m=0}^{N-1} h[m] x[((n-m))_N]$$

Circular convolution is the same as linear convolution if and only if $N \geq L + M - 1$.

Aliasing in time and frequency

Properties of the DFT:

- ③ Periodicity: $X[k + N] = X[k]$, $x[n + N] = x[n]$
- ① Conjugate symmetric: $x[n]$ Real $\leftrightarrow X[k] = X^*[-k]$
- ② Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z[k] = aX[k] + bY[k]$$

- ③ Time Shift: $x[n - n_0] \leftrightarrow e^{-j\frac{2\pi kn_0}{N}} X[k]$, paying attention to the fact that $x[n]$ is periodic in time
- ④ Frequency Shift: $e^{j\frac{2\pi k_0 n}{N}} x[n] \leftrightarrow X[k - k_0]$, paying attention to the fact that $X[k]$ is periodic in frequency

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Sample problems

If there's time, at this point in the lecture, I will do some sample problems from the practice exam. If there are problems whose solution you'd like to see, please ask.