Lecture 13: Temporal and Spatial Aliasing

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ECE 401: Signal and Image Analysis

1. Review: DFT
2. Sampled in Frequency $\leftrightarrow$ Periodic in Time
3. Phase Unwrapping
4. MRI
5. Summary

Outline

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Review: DFT

The DFT (discrete Fourier transform) of any signal is $X[k]$, given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$
$$x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F[k] = 1$$
$$g[n] = \delta\left[((n-n_0))_N\right] \leftrightarrow G[k] = e^{-j\frac{2\pi kn_0}{N}}$$

Properties of the DTFT

Properties worth knowing include:

0. Periodicity: $X[k+N] = X[k]$
1. Conjugate symmetric: $x[n]$ Real $\leftrightarrow X[k] = X^*[-k]$
2. Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z[k] = aX[k] + bY[k]$$

3. Time Shift: $x[n-n_0] \leftrightarrow e^{-j\frac{2\pi k n_0}{N}} X[k]$, paying attention to the fact that $x[n]$ is periodic in time
4. Frequency Shift: $e^{j\frac{2\pi k_0 n}{N}} x[n] \leftrightarrow X[k-k_0]$, paying attention to the fact that $X[k]$ is periodic in frequency

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Try the quiz!

Today's quiz is another circular convolution problem, because that might be the best way to think about periodicity in time. Go to the course web page and try it!

$x[n]$ periodic, $X[k] = NX_k$

If $x[n] = x[n+N]$, then its Fourier series is

$$X_k = \frac{1}{N}\sum_{n=1}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}},$$

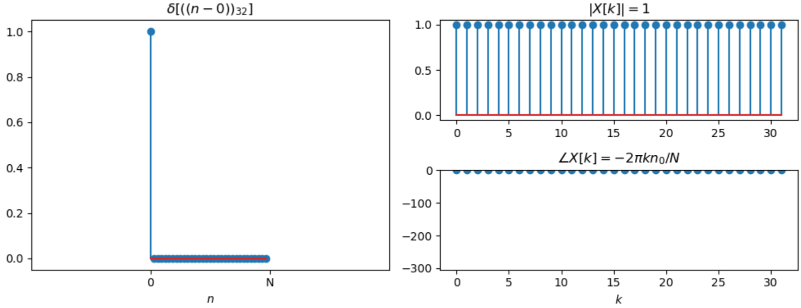
and its DFT is

$$X[k] = \sum_{n=1}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}$$

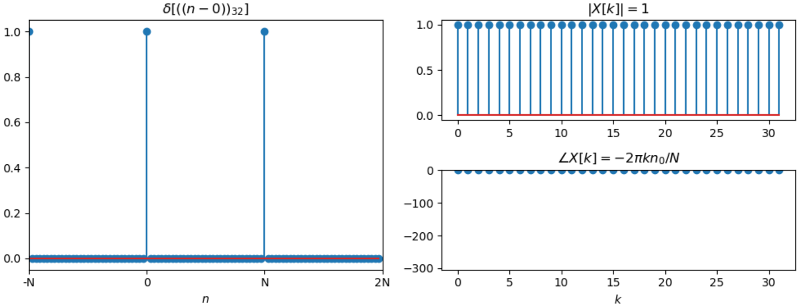
Delayed impulse wraps around

$$\delta\left[((n-n_0))_N\right] \leftrightarrow e^{-j\frac{2\pi k n_0}{N}}$$



Delayed impulse is really periodic impulse

$$\delta\left[((n-n_0))_N\right] \leftrightarrow e^{-j\frac{2\pi k n_0}{N}}$$



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Principal Phase

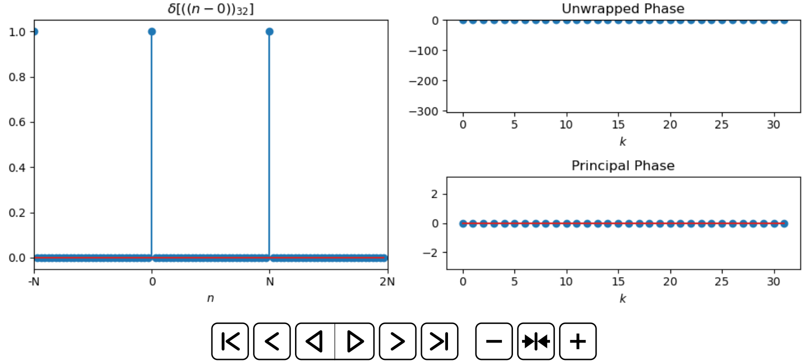
- Something weird going on: how can the phase keep getting bigger and bigger, but the signal wraps around?
- It's because the phase wraps around too!

$$\measuredangle X[k] = -\omega_k(N+n) = -\omega_k n, \quad \text{because } \omega_k = \frac{2\pi k}{N}$$

- **Principal phase =** add $\pm 2\pi$ to the phase, as necessary, so that $-\pi < \measuredangle X[k] \leq \pi$
- **Unwrapped phase =** let the phase be as large as necessary so that it is plotted as a smooth function of ω

Unwrapped phase vs. Principal phase

$$\delta\left[((n-n_0))_N\right] \leftrightarrow e^{-j\frac{2\pi k n_0}{N}}$$



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What is MRI?

To learn a little about spatial aliasing in MRI, let's start the first section of MP4.

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Summary

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1. Conjugate symmetric: $x[n]$ Real $\leftrightarrow X[k] = X^*[-k]$
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$$z[n] = ax[n] + by[n] \leftrightarrow Z[k] = aX[k] + bY[k]$$

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