Lecture 9: Discrete Fourier Transform

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ECE 401: Signal and Image Analysis

Outline

Review: Fourier Series

Every periodic continuous-time signal can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

The Fourier series coefficients can be computed as

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

Discrete-Time Fourier Series

The discrete-time Fourier series (DTFS) works the same way, but with a finite number of harmonics. Any periodic signal, $x[n+N]=x[n]$, can be written as

$$x[n]=\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

The coefficients are computed using a sum instead of an integral:

$$X_k=\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

Discrete Fourier Transform

Cooley and Tukey (1965), "An algorithm for the machine calculation of complex Fourier series," moved the $1/N$ factor to the synthesis equation, creating what we now call the **Discrete Fourier Transform** or DFT:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

Notice that $X[k] = NX_k$. DFT and DTFS are the same thing except scaling.

Outline

Periodic in Frequency

Unlike the continuous-time Fourier series (CTFS), the DTFS is periodic in frequency, with period N:

$$\begin{aligned} X[k+N] &= \sum_n x[n]e^{-j\frac{2\pi(k+N)n}{N}} \\ &= \sum_n x[n]e^{-j\frac{2\pi kn}{N}}e^{-j\frac{2\pi Nn}{N}} \\ &= \sum_n x[n]e^{-j\frac{2\pi kn}{N}} \\ &= X[k] \end{aligned}$$

Linearity

$$a x_1[n] + b x_2[n] \quad \leftrightarrow \quad a X_1[k] + b X_2[k]$$

Conjugate Symmetry of both CTFS and DTFS

If $x(t)$ is real-valued, then $X_k = X^*_{-k}$. Similarly, if $x[n]$ is real-valued, then $X[k] = X^*[-k]$:

$$\begin{aligned}
X[-k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi(-k)n}{N}} \\
&= \sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi kn}{N}} \\
&= \left(\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \right)^*
\end{aligned}$$

Frequency Shift

If a signal $w[n]$ is multiplied by any harmonic, its DFT is shifted in frequency:

$$x[n] = w[n]e^{j\frac{2\pi\ell n}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} w[n]e^{j\frac{2\pi\ell n}{N}} e^{-j\frac{2\pi kn}{N}}$$

$$= W[k-\ell]$$

Time Shift

Similarly, shifting in time (remember, $x[n]$ is periodic with period N!) multiplies the DFT by a phase shift:

$$
\begin{aligned}
x[n] &= w[n-n_0] \\
X[k] &= \sum_{n=0}^{N-1} w[n-n_0] e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{m=-n_0}^{N-n_0-1} w[m] e^{-j\frac{2\pi k(m+n_0)}{N}} \\
&= e^{-j\frac{2\pi kn_0}{N}} W[k]
\end{aligned}
$$

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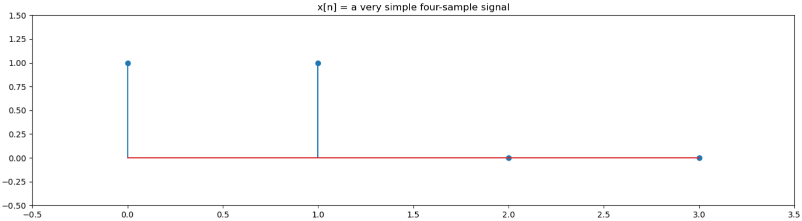
Example



Consider the signal

$$x[n] = \begin{cases} 1 & \text{n=0,1} \\ 0 & \text{n=2,3} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Example DFT

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi kn}{4}} \\
&= 1 + e^{-j\frac{2\pi k}{4}} \\
&= \begin{cases} 2 & k = 0 \\ 1 - j & k = 1 \\ 0 & k = 2 \\ 1 + j & k = 3 \end{cases}
\end{aligned}
$$

Notice the conjugate symmetry! $X[1]$ and $X[N-1]$ are complex conjugates. $X[0]$ and $X[N/2]$ are both real.

Example IDFT

$$X[k] = [2, (1-j), 0, (1+j)]$$

$$\begin{aligned}
x[n] &= \frac{1}{4}\sum_{k=0}^{3} X[k] e^{j\frac{2\pi kn}{4}} \\
&= \frac{1}{4}\left(2 + (1-j)e^{j\frac{2\pi n}{4}} + (1+j)e^{j\frac{6\pi n}{4}}\right) \\
&= \frac{1}{4}\left(2 + (1-j)j^n + (1+j)(-j)^n\right) \\
&= \begin{cases} 1 & n = 0, 1 \\ 0 & n = 2, 3 \end{cases}
\end{aligned}$$

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DFT of a Delta Function

Suppose $x[n]$ is a delta function:

$$x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \\ &= x[0]e^0 \\ &= 1 \end{aligned}$$

The DFT of a delta function is $X[k] = 1$!

DTFS of a Delta Function

Remember that we are assuming that $x[n]$ is periodic. When we say it's a delta function, we really mean that it's a sequence of delta functions — what we sometimes call an "impulse train:"

$$x[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN] = \begin{cases} 1 & n = mT_0, \quad m = \text{integer} \\ 0 & \text{otherwise} \end{cases}$$

Since the DFT of $\delta[n]$ is $X[k] = 1$, its Fourier series is

$$X_k = \frac{1}{N} X[k] = \frac{1}{N}$$

Therefore we can write it as a sum of harmonics that all have the same amplitude.

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi kn}{N}}$$

Shifted Delta Function

Suppose that we shift the delta function to the right by n_0 samples:

$$x[n] = \delta[n - n_0]$$

Applying the time-shift property of the DFT, we get

$$\begin{aligned} X[k] &= e^{-j\frac{2\pi k n_0}{N}} \text{DFT}\left(\delta[n]\right) \\ &= e^{-j\frac{2\pi k n_0}{N}} \end{aligned}$$

Quiz

Go to the course webpage, and try today's quiz!

Outline

Complex Exponential

Suppose $x[n] = e^{j\omega_0 n}$ for $0 \leq n < N$. As last time, there are two different cases:

- If $\omega_0 N =$integer, then $x[n]$ is continuous in time
- Otherwise, there's a discontinuity between $x[N-1]$ and $x[N]$.

DFT of a Complex Exponential

$$X[k] = \sum_{n=0}^{N-1} e^{j\omega n} e^{-j\frac{2\pi kn}{N}}$$

There is a geometric series formula that we will use many times in this class: $\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$. Using that formula, we get

$$X[k] = \frac{1 - e^{-j2\pi(k-\omega_0 N)}}{1 - e^{-j\frac{2\pi(k-\omega_0 N)}{N}}}$$

DFT of a Cosine

Suppose $x[n] = \cos(\omega_0 n)$ for $0 \le n < N$, then

$$X[k] = 0.5\frac{1 - e^{-j2\pi(k-\omega_0 N)}}{1 - e^{-j\frac{2\pi(k-\omega_0 N)}{N}}} + 0.5\frac{1 - e^{-j2\pi(k+\omega_0 N)}}{1 - e^{-j\frac{2\pi(k+\omega_0 N)}{N}}}$$

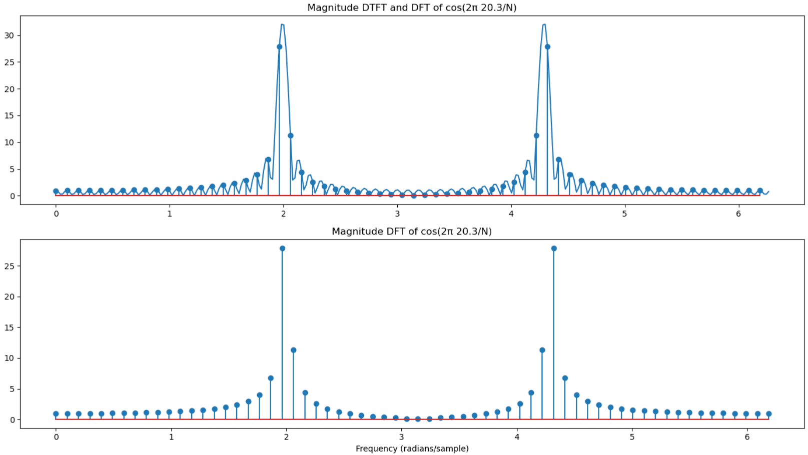
- If $\omega_0 N$ is an integer, then

$$X[k] = \begin{cases} \frac{N}{2} & k = \pm\omega_0 N \\ 0 & \text{otherwise} \end{cases}$$

- If $\omega_0 N$ is not an integer, then $X[k]$ is complicated.

DFT of a Cosine

$$x[n] = \cos\left(\frac{2\pi 20.3}{N} n\right), \quad 0 \le n \le 63$$



Outline

DFT Properties

1. **Periodic in Time and Frequency:**

$$x[n] = x[n+N], \qquad X[k] = X[k+N]$$

2. **Linearity:**

$$ax_1[n] + bx_2[n] \quad \leftrightarrow \quad aX_1[k] + bX_2[k]$$

3. **Time & Frequency Shift (Subject to periodicity):**

$$x[n]e^{j\frac{2\pi k_0 n}{N}} \quad \leftrightarrow \quad X[k-k_0]$$

$$x[n-n_0] \quad \leftrightarrow \quad X[k]e^{-j\frac{2\pi k n_0}{N}}$$