Lecture 5: Fourier Analysis

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ECE 401: Signal and Image Analysis

1. Review: Fourier Series
2. Computing the Fourier Series Coefficients
3. Calculation Examples
4. Visualization Examples
5. Summary

Outline

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Fourier Series

Any periodic signal, $x(t+T_0) = x(t)$, can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

In discrete time, you only need T_0 harmonics, because the period only has T_0 samples:

$$x[n] = \sum_{k=0}^{T_0-1} X_k e^{j2\pi nt/T_0}$$

Analysis and Synthesis

- **Fourier Synthesis** is the process of generating the signal, $x(t)$, given its spectrum. Last lecture, you learned how to do this, in general.
- **Fourier Analysis** is the process of finding the spectrum, X_k, given the signal $x(t)$. I'll tell you how to do that today.

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Fourier Analysis

Fourier said that any periodic signal, with a period of T_0 seconds, can be written

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

What I want to do, now, is to prove that if you know $x(t)$, you can find its Fourier series coefficients using the following formula:

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

Important note: the integral can be over any complete period. We usually write it from 0 to T_0, but it can be from $-\frac{T_0}{2}$ to $\frac{T_0}{2}$, or any other period.

Proof of Fourier Analysis: Step 1

$$
\begin{aligned}
&\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt\\
&=\frac{1}{T_0}\int_0^{T_0}\left(\sum_{\ell=-\infty}^{\infty} X_\ell e^{j2\pi\ell t/T_0}\right)e^{-j2\pi kt/T_0}dt\\
&=\frac{1}{T_0}\sum_{\ell=-\infty}^{\infty}\int_0^{T_0} X_\ell e^{j2\pi(\ell-k)t/T_0}dt
\end{aligned}
$$

Now we need to find out what is $\int_0^{T_0} X_\ell e^{j2\pi(\ell-k)t/T_0}dt$.

Proof of Fourier Analysis: Step 2

$$
\begin{aligned}
&\int_0^{T_0} X_\ell e^{j2\pi(\ell-k)t/T_0} dt \\
&= \int_0^{T_0} X_\ell \cos\left(\frac{2\pi(\ell-k)t}{T_0}\right) dt + j\int_0^{T_0} X_\ell \sin\left(\frac{2\pi(\ell-k)t}{T_0}\right) dt \\
&= \begin{cases} 0 & \ell \neq k \\ \int_0^{T_0} X_\ell dt = T_0 X_\ell = T_0 X_k & \ell = k \end{cases}
\end{aligned}
$$

Proof of Fourier Analysis: Step 3

$$
\begin{aligned}
&\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt\\
&=\frac{1}{T_0}\int_0^{T_0}\left(\sum_{\ell=-\infty}^{\infty} X_\ell e^{j2\pi \ell t/T_0}\right)e^{-j2\pi kt/T_0}dt\\
&=\frac{1}{T_0}\sum_{\ell=-\infty}^{\infty}\int_0^{T_0} X_\ell e^{j2\pi(\ell-k)t/T_0}dt\\
&=\frac{1}{T_0}\times\left(\cdots+0+0+0+T_0X_k+0+0+\cdots\right)\\
&=X_k
\end{aligned}
$$

Fourier Series

- **Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

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Square wave example

Suppose that

$$x(t) = \begin{cases} 1 & 0 \leq t < T_1 \\ 0 & T_1 \leq t < T_0 \end{cases}$$

What is X_k?

Square wave example

$$
\begin{aligned}
X_k &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\frac{2\pi kt}{T_0}}\,dt \\
&= \frac{1}{T_0}\int_0^{T_1} e^{-j\frac{2\pi kt}{T_0}}\,dt \\
&= \begin{cases} \frac{T_1}{T_0} & k=0 \\ \frac{1}{-j2\pi k}\left(e^{-j\frac{2\pi kT_1}{T_0}}-1\right) & k=1 \end{cases}
\end{aligned}
$$

Cosine example

Suppose that $x(t) = \cos(2\pi f_1 t)$ for $0 \leq t < T_0$. Notice that there are two very different cases:

- If $f_1 T_0$ is not an integer, then $x(t)$ has a discontinuity:
  - As t approaches T_0, $x(t)$ approaches $\cos(2\pi f_1 T_0)$.
  - When t reaches T_0, $x(t)$ suddenly resets. $x(t)$ must be periodic, so $x(T_0) = x(0) = 1$.
- If $f_1 T_0$ is an integer, then $x(t)$ is continuous.

Cosine example when $f_1 T_0$ is not an integer

$$
\begin{aligned}
X_k &= \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j\frac{2\pi kt}{T_0}}dt \\
&= \frac{1}{T_0}\int_0^{T_0} \cos(2\pi f_1 t) e^{-j\frac{2\pi kt}{T_0}}dt \\
&= \frac{1}{T_0}\int_0^{T_0} \frac{1}{2}\left(e^{j2\pi f_1 t} + e^{-j2\pi f_1 t}\right) e^{-j\frac{2\pi kt}{T_0}}dt \\
&= \frac{1}{2T_0}\left(\int_0^{T_0} e^{-j\frac{2\pi(k-f_1T_0)t}{T_0}}dt + \int_0^{T_0} e^{-j\frac{2\pi(k+f_1T_0)t}{T_0}}dt\right) \\
&= \frac{1}{-j4\pi(k-f_1T_0)}\left(e^{-j2\pi(k-f_1T_0)} - 1\right) \\
&+ \frac{1}{-j4\pi(k+f_1T_0)}\left(e^{-j2\pi(k+f_1T_0)} - 1\right)
\end{aligned}
$$

Cosine example when $f_1 T_0$ is an integer

$$X_k = \frac{1}{2T_0}\left(\int_0^{T_0} e^{-j\frac{2\pi(k-f_1 T_0)t}{T_0}} dt + \int_0^{T_0} e^{-j\frac{2\pi(k+f_1 T_0)t}{T_0}} dt\right)$$

$$= \begin{cases} \frac{1}{2} & k \pm f_1 T_0 = 0 \\ 0 & k \pm f_1 T_0 = \text{any other integer} \end{cases}$$

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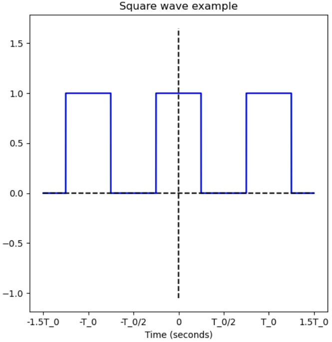
Another Square Wave

Let's try this square wave:

$$x(t) = \begin{cases} 1 & -\frac{T_0}{4} \le t \le \frac{T_0}{4} \\ 0 & \text{otherwise} \end{cases}$$



Square wave: the X_0 term

$$X_0 = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0}dt$$

...but if $k = 0$, then $e^{-j2\pi kt/T_0} = 1$!!!

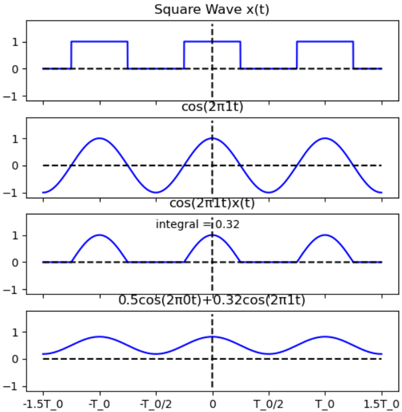
$$X_0 = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} 1dt = \frac{1}{2}$$

Square wave: the X_k terms, $k \neq 0$

$$
\begin{aligned}
X_k &= \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0} dt \\
&= \frac{1}{T_0} \left(\frac{1}{-j2\pi k/T_0} \right) \left[e^{-j2\pi kt/T_0} \right]_{-T_0/4}^{T_0/4} \\
&= \left(\frac{j}{2\pi k} \right) \left(e^{-j\pi k/2} - e^{j\pi k/2} \right) \\
&= \frac{1}{\pi k} \sin\left(\frac{\pi k}{2} \right) \\
&= \begin{cases} 0 & k \text{ even} \\ \pm\frac{1}{\pi k} & k \text{ odd} \end{cases}
\end{aligned}
$$

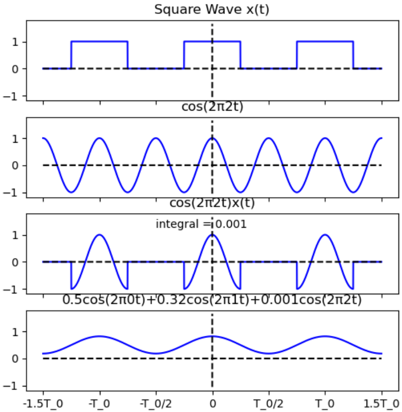
Square wave: the X_1 terms

$$X_1 = \frac{1}{\pi}$$



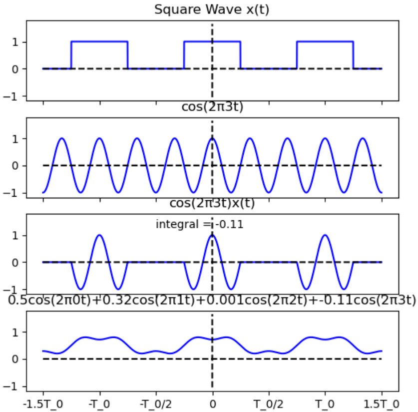
Square wave: the X_2 term

$$X_2 = 0$$



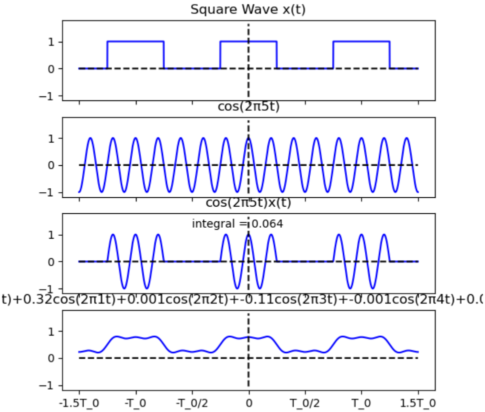
Square wave: the X_3 term

$$X_3 = -\frac{1}{3\pi}$$



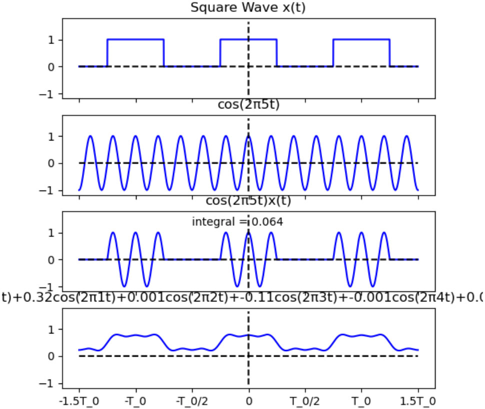
Square wave: the X_5 term

$$X_5 = \frac{1}{5\pi}$$



Square wave: the whole Fourier series

$$x(t) = \frac{1}{2} + \frac{1}{\pi}\left(\cos\left(\frac{2\pi t}{T_0}\right) - \frac{1}{3}\cos\left(\frac{6\pi t}{T_0}\right) + \frac{1}{5}\cos\left(\frac{10\pi t}{T_0}\right) - \frac{1}{7}\dots\right)$$



Quiz

Try the quiz! Go to the course webpage, and click on today's date.

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Summary

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$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$