Lecture 5: Sampling and Aliasing

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis

Outline

How to sample a continuous-time signal

Suppose you have some continuous-time signal, $x(t)$, and you'd like to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{F_s}$ seconds:

$$x[n] = x(t = nT_s)$$

For example, suppose $x(t) = \sin(2\pi 1000t)$. By sampling at $F_s = 16000$ samples/second, we get

$$x[n] = \sin\left(2\pi 1000 \frac{n}{16000}\right) = \sin(\pi n/8)$$

Not every sine wave can be reconstructed from its samples

For example, two signals $x_1(t)$ and $x_2(t)$, at 10kHz and 6kHz respectively:

$$x_1(t) = \cos(2\pi 10000 t), \quad x_2(t) = \cos(2\pi 6000 t)$$

Let's sample them at $F_s = 16,000$ samples/second:

$$x_1[n] = \cos\left(2\pi 10000 \frac{n}{16000}\right), \quad x_2[n] = \cos\left(2\pi 6000 \frac{n}{16000}\right)$$

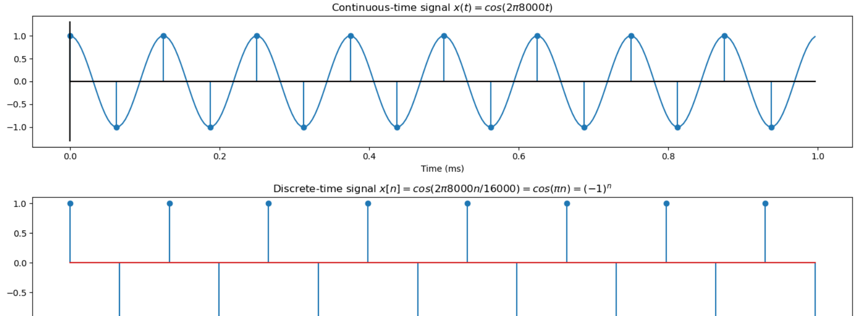
Simplifying a bit, we discover that $x_1[n] = x_2[n]$. We say that the 10kHz tone has been "aliased" to 6kHz:

$$x_1[n] = \cos\left(\frac{5\pi n}{4}\right) = \cos\left(\frac{3\pi n}{4}\right)$$
$$x_2[n] = \cos\left(\frac{3\pi n}{4}\right) = \cos\left(\frac{5\pi n}{4}\right)$$

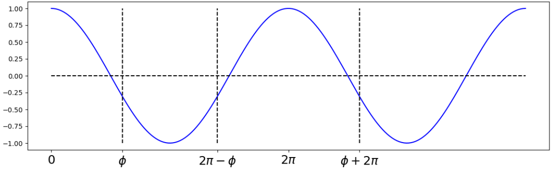
What is the highest frequency that can be reconstructed?

The minimum sampling rate that avoids aliasing is called the **Nyquist rate,** and it is $F_s = 2f$. Conversely, we talk about the the **Nyquist frequency,** $F_N = F_S/2$, which is the highest frequency pure tone that can be reconstructed at sampling rate F_s. If $x(t) = \cos(2\pi F_N t)$, then

$$x[n] = \cos\left(2\pi F_N \frac{n}{F_S}\right) = \cos(\pi n) = (-1)^n$$



Aliased Frequency

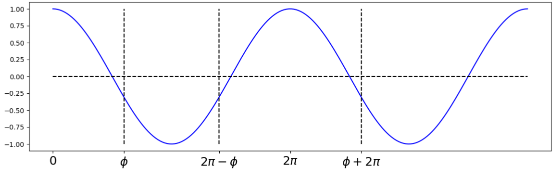


Aliasing comes from two sources:

$$\cos(\phi) = \cos(2\pi n - \phi)$$
$$\cos(\phi) = \cos(\phi - 2\pi n)$$

The equations above are true for any integer n.

Aliased Frequency



Let's plug in $\phi = \frac{2\pi fn}{F_s}$, and $2\pi = \frac{2\pi F_s}{F_s}$. That gives us:

$$\cos\left(\frac{2\pi fn}{F_s}\right) = \cos\left(\frac{2\pi n(F_s - f)}{F_s}\right)$$

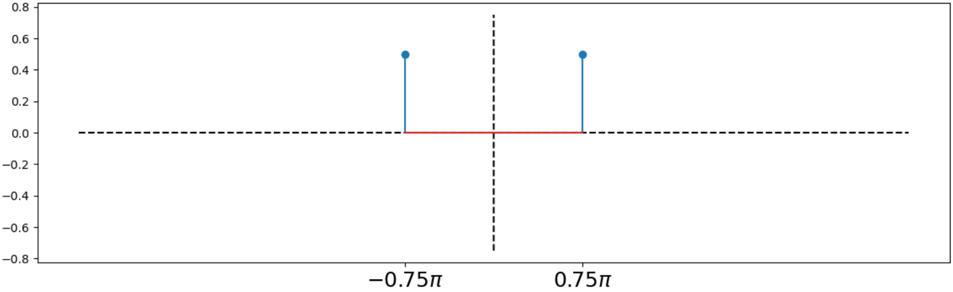
$$\cos\left(\frac{2\pi fn}{F_s}\right) = \cos\left(\frac{2\pi(f - F_s)n}{F_s}\right)$$

So a discrete-time cosine at frequency f is also a cosine at frequency $F_s - f$, and it's also a cosine at $f - F_s$.

Spectrum of a Continuous-time Cosine

A continuous-time cosine is the sum of two complex exponentials:

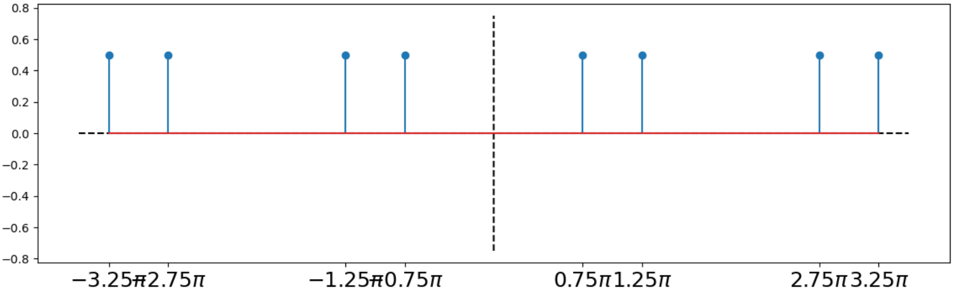
$$\cos(0.75\pi n) = \frac{1}{2}e^{j0.75\pi n} + \frac{1}{2}e^{-j0.75\pi n}$$



Spectrum of a Discrete-time Cosine

A discrete-time cosine is **still** just the sum of two complex exponentials, but each of those two complex exponentials is identical to an exponential at any other multiple of 2π:

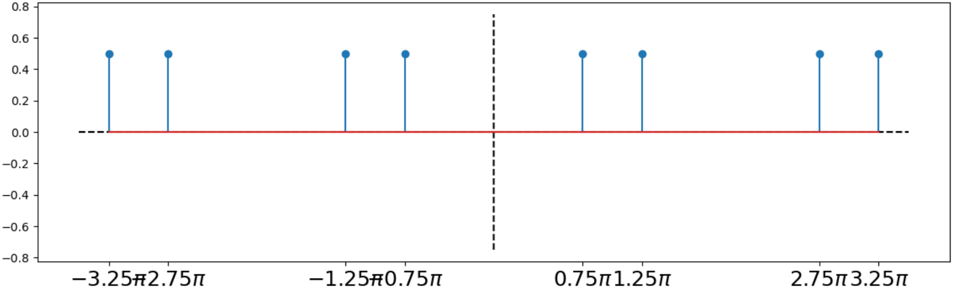
$$0.75\pi + \text{integer multiples of } 2\pi = \{-3.25\pi, -1.25\pi, 0.75\pi, 2.75\pi\}$$
$$\text{integer multiples of } 2\pi - 0.75\pi = \{-2.75\pi, -0.75\pi, 1.25\pi, 3.25\pi\}$$



Spectrum of an Aliased Discrete-time Cosine

Now consider what happens as we lower F_s. As we lower F_s, the frequency $\omega = \frac{2\pi f}{F_s}$ gets higher and higher, until aliasing occurs:

$$\cos(\omega n) = \cos((2\pi - \omega)n)$$

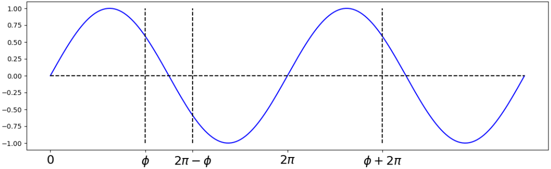


Aliased Frequency

- A discrete-time cosine at frequency f is also a cosine at frequency $F_s - f$, and it's also a cosine at $f - F_s$.
- So which of those frequencies will we hear when we play the sinusoid back again?
- **ANSWER:** any frequency that can be reconstructed by the analog-to-digital converter. That means any frequency below the Nyquist frequency, $F_N = F_s/2$.

Outline

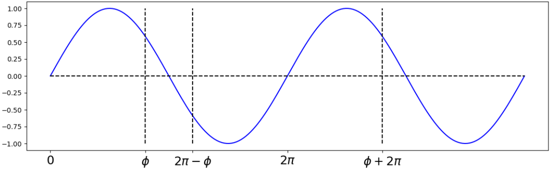
Sine is Different



Sine waves are different for the following reason:

$$\sin(\phi) = -\sin(2\pi n - \phi)$$
$$\sin(\phi) = \sin(\phi - 2\pi n)$$

Sine is Different



Therefore:

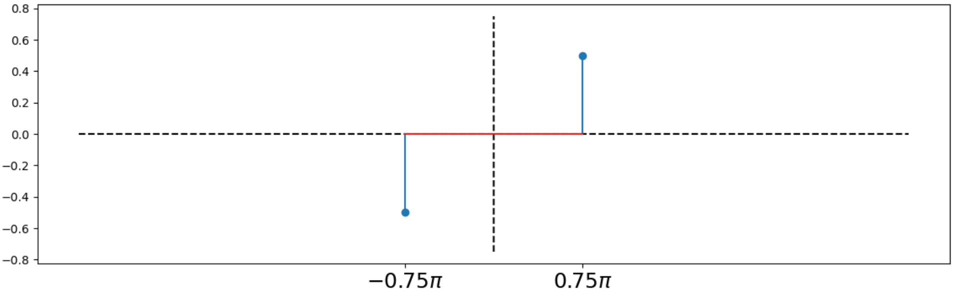
$$\sin\left(\frac{2\pi fn}{F_s}\right) = -\sin\left(\frac{2\pi n(F_s - f)}{F_s}\right)$$
$$\sin\left(\frac{2\pi fn}{F_s}\right) = \sin\left(\frac{2\pi(f - F_s)n}{F_s}\right)$$

So a discrete-time sine at frequency f is also a **negative** sine at frequency $F_s - f$, and a **positive** sine at frequency $f - F_s$.

Spectrum of a Continuous-time Sine

A continuous-time sine is the **difference** of two complex exponentials:

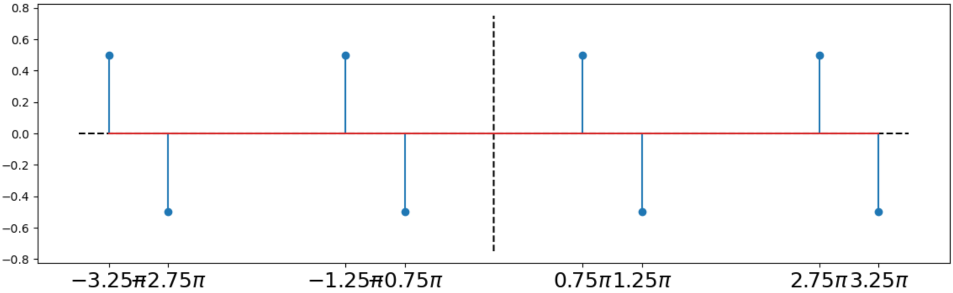
$$\sin(0.75\pi n) = \frac{1}{2j}e^{j0.75\pi n} - \frac{1}{2j}e^{-j0.75\pi n}$$



Spectrum of a Discrete-time Sine

A discrete-time sine is still just the difference of two complex exponentials, but each of those two complex exponentials is identical to an exponential at any other multiple of 2π:

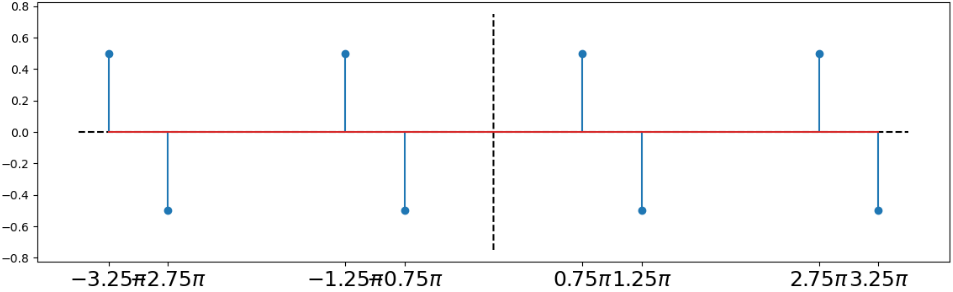
$$e^{j2.75\pi n} = e^{-j2\pi n}e^{j0.75\pi n} = e^{j0.75\pi n}$$



Spectrum of an Aliased Discrete-time Sine

Now consider what happens as we lower F_s. As we lower F_s, the frequency $\omega = \frac{2\pi f}{F_s}$ gets higher and higher, until aliasing occurs:

$$\sin(\omega n) = -\sin((2\pi - \omega)n)$$



Aliased Phase of a General Phasor

For a general complex exponential, we get:

$$ze^{j\phi} = ze^{j(\phi - 2\pi n)} = \left(z^* e^{j(2\pi n - \phi)}\right)^*$$

Therefore:

$$\Re\left\{ze^{j\frac{2\pi fn}{F_s}}\right\} = \Re\left\{ze^{j\frac{2\pi(f-F_s)n}{F_s}}\right\} = \Re\left\{z^* e^{j\frac{2\pi(F_s-f)n}{F_s}}\right\}$$

Aliased Phase of a General Phasor

Suppose we have some frequency f, and we're trying to find its aliased frequency f_a.

- Among the several possibilities, if $f_a = F_s - f$ is below Nyquist, then that's the frequency we'll hear. Its phasor will be the complex conjugate of the original phasor,

$$z_a = z^*$$

- On the other hand, if $f_a = f - F_s$ is below Nyquist, then that's the frequency we'll hear. Its phasor will be the same as the phasor of the original sinusoid:

$$z_a = z$$

Try the Quiz!

Go to the course webpage, and try today's quiz!

Outline

Summary

- A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, $f < \frac{F_s}{2}$.
- If the Nyquist criterion is violated, then:
  - If $\frac{F_s}{2} < f < F_s$, then it will be aliased to

$$f_a = F_s - f$$
$$z_a = z^*$$

    i.e., the sign of all sines will be reversed.
  - If $F_s < f < \frac{3F_s}{2}$, then it will be aliased to

$$f_a = f - F_s$$
$$z_a = z$$