UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal Processing
PRACTICE EXAM 3

Exam will be December 4, 2026, in class

- This practice exam is about 6 times the expected length of the actual exam.
- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not leave any unresolved integrals or summations in your answer. Other than that, however, you should not simplify explicit numerical expressions.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

NetID: _____

DTFT

$$X(\omega) = \sum_n x[n] e^{-j\omega n}$$

$$x[n] = e^{j\theta n} \leftrightarrow X(\omega) = 2\pi\delta(\omega - \theta)$$

$$x[n] = \left(\frac{\omega_c}{\pi}\right) \mathrm{sinc}(\omega_c n) \leftrightarrow X(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} 1 & |n| \leq M \\ 0 & \text{otherwise} \end{cases} \leftrightarrow X(\omega) = \frac{\sin(\omega(2M+1)/2)}{\sin(\omega/2)}$$

Z Transform Pairs

$$X(z) = \sum_n x[n] z^{-n}$$

$$x[n] = a^n u[n] \leftrightarrow X(z) = \frac{1}{1 - az^{-1}}$$

$$x[n] = \delta[n-k] \leftrightarrow X(z) = z^{-k}$$

$$e^{-\sigma_1 n} \frac{\sin\left(\omega_1(n+1)\right)}{\sin(\omega_1)} u[n] \leftrightarrow \frac{1}{(1 - e^{-\sigma_1 - j\omega_1} z^{-1})(1 - e^{-\sigma_1 + j\omega_1} z^{-1})}$$

Notch Filter

$$H(z) = \frac{(1 - e^{j\omega_1} z^{-1})(1 - e^{-j\omega_1} z^{-1})}{(1 - ae^{j\omega_1} z^{-1})(1 - ae^{-j\omega_1} z^{-1})}$$

$$\mathrm{BW} = -2\ln(a)$$

1. (10 points) You have recorded an electrocardiogram signal, $x[n]$, with a sampling frequency of $F_s = 1.2kHz$. Unfortunately, it has been corrupted by power line noise: it has a big sinusoidal component at 60Hz. Fortunately, you know how to eliminate power line noise using a notch filter. All you have to do is to pass the signal through a difference equation:

$$y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2] \tag{1}$$

Use a pole amplitude of 0.98. What are b_1, b_2, a_1, and a_2?

(Note: leave your answer in the form of an explicit numerical expression. For example, if you discover that $a_1 = (0.3)^2 \sin(2\pi/400)$, then you should leave it in that form instead of trying to simplify.)

Solution:

$$b_1 = -2\cos\left(\frac{2\pi 60}{1200}\right)$$
$$b_2 = 1$$
$$a_1 = -2 \times 0.98\cos\left(\frac{2\pi 60}{1200}\right)$$
$$a_2 = (0.98)^2$$

2. (10 points) Suppose $x[n]$ is a random signal with the following autocorrelation:

$$r_x[\tau] = \frac{1}{16}\text{sinc}^2\left(\frac{\pi n}{4}\right) = \left(\frac{\sin(\pi n/4)}{\pi n}\right)^2$$

Suppose $e[n] = x[n] - ax[n-1]$, and you want to find a in order to minimize $E\left[e^2[n]\right]$. Find the numerical value of a ("numerical" in the sense that there are no variables in your answer, however, your answer may include constants like π and $\sqrt{2}$ and/or trigonometric functions of such constants).

Solution:

$$\begin{aligned} a = \mathbf{R}^{-1}\gamma &= \frac{r_x[1]}{r_x[0]} \\ &= \frac{\text{sinc}^2(\pi/4)}{\text{sinc}^2(0)} \\ &= \left(\frac{4\sin(\pi/4)}{\pi}\right)^2 \end{aligned}$$

3. (20 points) Suppose you have a signal sampled at $F_s = 600$ samples/second. You wish to create a notch filter to eliminate a noise component at 60Hz. You choose to do this using the following filter:

$$H(z) = \frac{(1 - r_1 z^{-1})(1 - r_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

(a) Specify the values of the poles p_1, p_2 and the zeros r_1, r_2. Note that there is one free parameter in your answer that is not specified by the problem statement; you may set that free parameter to any reasonable value.

Solution: $r_1 = e^{j\pi/5}$, $r_2 = e^{-j\pi/5}$, $p_1 = 0.99e^{j\pi/5}$, $p_2 = 0.99e^{-j\pi/5}$

(b) Now suppose you are given a system function

$$H(z) = \frac{(1 - r_1 z^{-1})(1 - r_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

and you wish to implement this using the equation

$$y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

Find b_1, b_2, a_1 and a_2 in terms of r_1, r_2, p_1 and p_2.

Solution: $b_1 = -(r_1 + r_2)$, $b_2 = r_1 r_2$, $a_1 = -(p_1 + p_2)$, $a_2 = p_1 p_2$

4. (20 points) $x[n]$ is a white noise signal with energy $r_x[0] = 25$. $y[n]$ is a filtered signal, computed as:

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

Find $r_y[\tau]$, the autocorrelation of $y[n]$.

Solution: Since $x[n]$ is white noise,

$$r_x[n] = 25\delta[n]$$

Therefore

$$\begin{aligned} r_y[\tau] &= \sum_m y[m]y[m-\tau] \\ &= \frac{1}{9}\sum_m \left(x[m] + x[m-1] + x[m-2]\right)\left(x[m-\tau] + x[m-1-\tau] + x[m-2-\tau]\right) \\ &= \begin{cases} \frac{3}{9}r_x[0] & \tau = 0 \\ \frac{2}{9}r_x[0] & = \tau = \pm 1 \\ \frac{1}{9}r_x[0] & = \tau = \pm 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{75}{9} & \tau = 0 \\ \frac{50}{9} & = \tau = \pm 1 \\ \frac{25}{9} & = \tau = \pm 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

5. (20 points) Consider the following IIR filter:

$$y[n] = x[n] + x[n-1] - 0.5y[n-1] \tag{2}$$

(a) Calculate the impulse response of this system, $h[n]$.

Solution:

$$h[n] = (-0.5)^n u[n] + (-0.5)^{n-1} u[n-1]$$

(b) Calculate the system response, $H(z)$, using any method that you like.

Solution:

$$H*(z) = \frac{1 + z^{-1}}{1 + 0.5z^{-1}}$$

(c) Use the relationship between Z-transform and DTFT to find the magnitude frequency response, $|H(\omega)|$. Plot $|H(\omega)|$ as a function of ω, for $0 \leq \omega \leq \pi$. Label the amplitude at $\omega = 0$, $\omega = \frac{\pi}{2}$, and at $\omega = \pi$.

Solution:

$$\begin{aligned}
|H(\omega)| &= \left| \frac{1 + e^{-j\omega}}{1 + 0.5e^{-j\omega}} \right| \\
&= \sqrt{\frac{(1 + \cos\omega)^2 + \sin^2\omega}{(1 + 0.5\cos\omega)^2 + 0.25\sin^2\omega}} \\
&= \begin{cases} \sqrt{\frac{4}{(1.5)^2}} & \omega = 0 \\ \sqrt{\frac{2}{1.25}} & \omega = \frac{\pi}{2} \\ \sqrt{\frac{1}{0.5}} & \omega = \pi \end{cases}
\end{aligned}$$

6. (20 points) Suppose you have a signal $x[n] = s[n] + v[n]$ corrupted by a narrowband noise, $v[n]$, at the frequency $\frac{\pi}{4}$ radians/sample.

   (a) Specify the linear constant-coefficient difference equation for a filter that eliminates the noise without distorting the signal too much. You may choose any bandwidth for the filter that you find to be convenient.

   Solution:

   $$y[n] = x[n] - 2\cos\left(\frac{\pi}{4}\right)x[n-1] + x[n-2] + 2 \times 0.9\cos\left(\frac{\pi}{4}\right)y[n-1] - (0.9)^2 y[n-2]$$

   (where 0.9 may be replaced by any number between 0 and 1).

   (b) Sketch the magnitude frequency response of the filter you designed in part (a). Specify its value at these three frequencies: $\omega = 0$, $\omega = \frac{\pi}{4}$, and $\omega = \frac{\pi}{4} - \sigma$, where σ is the bandwidth you chose in part (a).

   Solution:

   $$|H(\omega)| = \begin{cases} 1 & \omega = 0 \\ 0 & \omega = \frac{\pi}{4} \\ \frac{1}{\sqrt{2}} & \omega = \frac{\pi}{4} - \sigma \end{cases}$$

7. (20 points) The vowel /æ/, as in "had," is produced by pushing your tongue down and forward. The resulting vocal tract shape has resonant frequencies of F1=900, F2=1700, F3=2700, F4=3500Hz, with bandwidths of roughly B1=100, B2=150, B3=250, and B4=300Hz, respectively.

(a) Assume a sampling rate of $F_s = 10,000$Hz. Express the four resonant frequencies, and their bandwidths, in units of radians/sample.

Solution:

$$\omega_1 = \frac{2\pi 900}{10000}, \quad \sigma_1 = \frac{\pi 100}{10000}$$
$$\omega_2 = \frac{2\pi 1700}{10000}, \quad \sigma_2 = \frac{\pi 150}{10000}$$
$$\omega_3 = \frac{2\pi 2700}{10000}, \quad \sigma_3 = \frac{\pi 250}{10000}$$
$$\omega_4 = \frac{2\pi 3500}{10000}, \quad \sigma_4 = \frac{\pi 300}{10000}$$

(b) This transfer function can be written as

$$H(z) = \frac{1}{\prod_{k=1}^{8}(1 - p_k z^{-1})}$$

Give the eight pole locations, p_1 through p_8.

Solution:

$$p_1 = e^{-\sigma_1 + j\omega_1}, \quad p_2 = e^{-\sigma_1 - j\omega_1}$$
$$p_3 = e^{-\sigma_2 + j\omega_2}, \quad p_4 = e^{-\sigma_2 - j\omega_2}$$
$$p_5 = e^{-\sigma_3 + j\omega_3}, \quad p_6 = e^{-\sigma_3 - j\omega_3}$$
$$p_7 = e^{-\sigma_4 + j\omega_4}, \quad p_8 = e^{-\sigma_4 - j\omega_4}$$

8. (10 points) Consider the filter

$$H(z) = \frac{1}{(1 - 0.9e^{j\pi/4}z^{-1})(1 - 0.9e^{-j\pi/4}z^{-1})}$$

Use partial fraction expansion to find $h[n]$. You may leave it in the form of a pair of complex exponentials; you don't need to simplify to sines and cosines.

Solution:

$$\begin{aligned}
\frac{1}{(1 - 0.9e^{j\pi/4}z^{-1})(1 - 0.9e^{-j\pi/4}z^{-1})} &= \frac{c_1}{1 - 0.9e^{j\pi/4}z^{-1}} + \frac{c_2}{1 - 0.9e^{-j\pi/4}z^{-1}} \\
1 &= c_1(1 - 0.9e^{-j\pi/4}z^{-1}) + c_2(1 - 0.9e^{j\pi/4}z^{-1}) \\
1 &= c_1 + c_2 \\
0 &= c_1e^{-j\pi/4} + c_2e^{j\pi/4}
\end{aligned}$$

The last two equations are jointly solved by

$$\begin{aligned}
c_1 &= \frac{1}{2}(1 - j) = \frac{\sqrt{2}}{2}e^{-j\pi/4} \\
c_2 &= \frac{1}{2}(1 + j) = \frac{\sqrt{2}}{2}e^{j\pi/4}
\end{aligned}$$

In terms of these constants, the impulse response is

$$h[n] = c_1(0.9e^{j\pi/4})^n u[n] + c_2(0.9e^{-j\pi/4})^n u[n]$$

9. (17 points) Consider the difference equation

$$y[n] = x[n] - 0.6x[n - 1] + 0.2x[n - 2]$$

Find explicit numerical expressions for the frequencies, $\omega_1 = \angle z_1$ and $\omega_2 = \angle z_2$, of the zeros of this filter.

Solution:

$$\begin{aligned}
\omega_1 &= \text{atan}\left(\frac{\sqrt{0.8 - 0.36}}{0.6}\right) \\
\omega_2 &= -\text{atan}\left(\frac{\sqrt{0.8 - 0.36}}{0.6}\right)
\end{aligned}$$

10. (16 points) Suppose you have designed a filter with the following system function:

$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 0.7z^{-1}}$$

What is $h[n]$, the impulse response of this filter?

Solution:

$$h[n] = (0.7)^n u[n] + 0.5(0.7)^{n-1} u[n-1]$$

11. (17 points) A recording of a conversation, at $F_s = 16,000$ samples/second, is corrupted by an additive pure tone at exactly $f_1 = 1000$Hz. The tone is quite narrowband, so you can get rid of it using a notch filter with a 20Hz bandwidth. Write the difference equation for this notch filter; specify all coefficients as numbers, or as explicit numerical expressions.

Solution:

$$y[n] = x[n] - 2\cos\left(\frac{\pi}{8}\right) x[n-1] + x[n-2] + 2e^{-\frac{\pi}{800}} \cos\left(\frac{\pi}{8}\right) y[n-1] - e^{-\frac{2\pi}{800}} y[n-2]$$

12. (17 points) A particular signal has the following z-transform:

$$X(z) = \frac{1}{1 - 0.9e^{0.1\pi j} z^{-1}}$$

The signal, $x[n]$, is complex-valued in the time domain, with an exponentially decreasing absolute value. Find an explicit numerical expression for the smallest integer, n, such that

$$\frac{|x[n]|}{|x[0]|} \leq \frac{1}{e},$$

where e is the base of the natural logarithm.

Solution:

$$n \geq -\frac{1}{\ln(0.9)}$$

13. (20 points) Suppose that

$$H(z) = \frac{1}{1 - 0.9e^{j\pi/6} z^{-1}} + \frac{1}{1 - 0.9e^{-j\pi/6} z^{-1}}$$

Find the pole(s) and zero(s) of $H(z)$.

Solution:

$$\begin{aligned} H(z) &= \frac{(1 - 0.9e^{j\pi/6} z^{-1}) + (1 - 0.9e^{-j\pi/6} z^{-1})}{(1 - 0.9e^{j\pi/6} z^{-1})(1 - 0.9e^{-j\pi/6} z^{-1})} \\ &= \frac{2 - 1.8\cos(\pi/6) z^{-1}}{(1 - 0.9e^{j\pi/6} z^{-1})(1 - 0.9e^{-j\pi/6} z^{-1})} \\ &= 2\frac{1 - 0.9\cos(\pi/6) z^{-1}}{(1 - 0.9e^{j\pi/6} z^{-1})(1 - 0.9e^{-j\pi/6} z^{-1})} \end{aligned}$$

This has one zero, r_1, and two poles, p_1 and p_2:

$$\begin{aligned} r_1 &= 0.9\cos(\pi/6) = 0.45\sqrt{3} \\ p_1 &= 0.9e^{j\pi/6} \\ p_2 &= 0.9e^{-j\pi/6} \end{aligned}$$

14. (20 points) Consider a notch filter with zeros at $r_1 = e^{j0.47\pi}$ and $r_2 = e^{-j0.47\pi}$, and with poles at $p_1 = 0.999e^{j0.47\pi}$ and $p_2 = 0.999e^{-j0.47\pi}$.

(a) What is the 3dB bandwidth of the notch, expressed in radians per sample?

Solution: The bandwidth is $2\sigma = -2\ln(0.999)$.

(b) This filter can be implemented as

$$y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

Find b_1, b_2, a_1, and a_2.

Solution:

$$\begin{aligned} H(z) &= \frac{(1 - e^{j0.47\pi}z^{-1})(1 - e^{-j0.47\pi}z^{-1})}{(1 - 0.999e^{j0.47\pi}z^{-1})(1 - 0.999e^{-j0.47\pi}z^{-1})} \\ &= \frac{1 - 2\cos(0.47\pi)z^{-1} + z^{-2}}{1 - 1.998\cos(0.47\pi)z^{-1} + (0.999)^2 z^{-2}} \\ &= \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned}$$

So,

$$\begin{aligned} b_1 &= -2\cos(0.47\pi) \\ b_2 &= 1 \\ a_1 &= -1.998\cos(0.47\pi) \\ a_2 &= (0.999)^2 \end{aligned}$$

15. (20 points) A particular bell has a resonance at 440Hz, with a decay time of half a second, and another resonance at 1320Hz, with a decay time of 3 seconds. Find a filter, $H(z)$, whose impulse response sounds like the impulse response of this bell if played through a D/A at $F_s = 10,000$ samples per second.

Solution: The decay time specifies the filter bandwidths; we have that $e^{-\sigma n} = e^{-1}$ when n =decay time×sampling rate. In this case,

$$\sigma_1 = \frac{1}{0.5 \times 10000} = \frac{1}{5000}$$
$$\sigma_2 = \frac{1}{3 \times 10000} = \frac{1}{30000}$$
$$\omega_1 = \frac{2\pi 440}{10000}$$
$$\omega_2 = \frac{2\pi 1320}{10000}$$

The filter can actually be either a parallel or a series connection of these two resonators. If it's a parallel connection, it would be

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} + \frac{1}{(1 - p_2 z^{-1})(1 - p_2^* z^{-1})}$$
$$p_1 = e^{-n/5000} e^{j\frac{2\pi 440}{10000}}$$
$$p_2 = e^{-n/30000} e^{j\frac{2\pi 1320}{10000}}$$

16. (22 points) Suppose $H(z)$ is given by

$$H(z) = \frac{1 - 0.9z^{-1}}{(1 - 0.4z^{-1})(1 + 0.4z^{-1})}$$

What is $h[n]$?

Solution:

$$h[n] = -\left(\frac{5}{8}\right)(0.4)^n u[n] + \left(\frac{13}{8}\right)(-0.4)^n u[n]$$

17. (22 points) Suppose you have a signal sampled at $F_s = 44,000$ samples/second, and you want to filter it to eliminate a narrowband noise signal at 1000Hz, with a 3dB bandwidth of 100Hz. Specify a filter $H(z)$ that will accomplish this task.

Solution:

$$H(z) = \frac{(1 - e^{j\frac{\pi}{22}} z^{-1})(1 - e^{-j\frac{\pi}{22}} z^{-1})}{(1 - e^{-\frac{\pi}{440} + j\frac{\pi}{22}} z^{-1})(1 - e^{-\frac{\pi}{440} - j\frac{\pi}{22}} z^{-1})}$$

18. (22 points) Consider the system

$$y[n] = x[n] + \frac{2}{3}x[n-1] + \frac{2}{3}y[n-1]$$

Plot a pole-zero plot for the system function $H(z) = Y(z)/X(z)$, showing the locations in the complex plane of every zero and every pole of $H(z)$.

Solution: The plot should show a complex plane (real and imaginary axes), with a pole at $z = \frac{2}{3}$, and a zero at $z = -\frac{2}{3}$.

19. (22 points) Consider the filter

$$y[n] = x[n] - a_1 y[n-1] - a_2 y[n-2]$$

The impulse response of this filter is $h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1(n+1))u[n]$. Use the quadratic formula ($\frac{-b \pm \sqrt{b^2-4ac}}{2a}$) to write explicit formulas for σ_1 and ω_1 in terms of a_1 and a_2. You may assume that $a_2 > a_1^2/4$.

Solution: The roots are

$$\begin{aligned} z_1, z_2 &= \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \\ &= e^{-\sigma_1 \pm j\omega_1} \end{aligned}$$

So

$$\sigma_1 = \Re\{\ln\left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}\right)\}$$

$$\omega_1 = \Im\{\ln\left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}\right)\}$$

Alternate solution:

$$\begin{aligned} z_1 z_2 &= e^{-2\sigma_1} \\ &= \left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}\right)\left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}\right) \\ &= \frac{1}{4}(a_1^2 - (a_1^2 - 4a_2)) = a_2 \\ &\Rightarrow \sigma_1 = -\frac{1}{2}\ln a_2 \end{aligned}$$

$$\begin{aligned} z_1 + z_2 &= 2e^{-\sigma_1}\cos(\omega_1) \\ &= -a_1 \\ \cos(\omega_1) &= -\frac{a_1}{2}e^{\sigma_1} \\ &= -\frac{a_1}{2\sqrt{a_2}} \\ \omega_1 &= \cos^{-1}\left(-\frac{a_1}{2\sqrt{a_2}}\right) \end{aligned}$$

20. (22 points) You have a signal sampled at $F_s = 10,000$ samples/second, corrupted by an annoying tone at 1000Hz. You would like to get rid of the 1000Hz tone, while leaving all components of the signal at frequencies greater than 1010Hz or less than 990Hz essentially unchanged. If you choose appropriate constants a, b, c, and d, you can accomplish this denoising task using the following system:

$$y[n] = x[n] + ax[n-1] + bx[n-2] + cy[n-1] + dy[n-2]$$

Choose constants a, b, c, and d such that the system above will get rid of the 1000Hz tone without much changing input components at frequencies above 1010Hz or below 990Hz.

Solution: The center frequency and bandwidth are

$$\omega_c = \frac{2\pi 1000}{10000} = 0.2\pi$$
$$-2\ln(a) = \frac{2\pi 20}{10000} = 0.004\pi$$

So the coefficients are

$$a = -2\cos(0.2\pi)$$
$$b = 1$$
$$c = 2e^{-0.002\pi}\cos(0.2\pi)$$
$$d = -e^{-0.004\pi}$$

21. (22 points) Suppose you have a second-order system whose system function is

$$H(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})}$$

You would like to find the impulse response $h[n]$ of this system. One way to do so is by finding coefficients c and d such that

$$H(z) = \frac{c}{1 - az^{-1}} + \frac{d}{1 - bz^{-1}} \leftrightarrow h[n] = (ca^n + db^n)u[n]$$

Find c and d in terms of a and b.

Solution:

$$c = \frac{1}{1 - b/a}$$

$$d = \frac{1}{1 - a/b}$$