UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal Processing
PRACTICE EXAM 2

Exam will be November 2, 2026, in class

- This practice exam is about 6 times the expected length of the actual exam.
- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not leave any unresolved integrals or summations in your answer. Other than that, however, you should not simplify explicit numerical expressions.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

NetID: _____

DTFT and DFT

$$h[n] * \cos(\omega n) = |H(\omega)| \cos(\omega n + \angle H(\omega))$$

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}, \quad h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} d\omega$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$

DTFT and DFT Properties

1. Periodicity: $X(\omega + 2\pi) = X(\omega)$, $X[k+N] = X[k]$
2. Linearity: $z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega), \quad Z[k] = aX[k] + bY[k]$
3. Time Shift: $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega), \quad e^{-j\frac{2\pi k n_0}{N}} X[k]$ if $x[n]$ periodic
4. Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0), \quad X\left[k - \frac{N\omega_0}{2\pi}\right]$ if $N\omega_0/2\pi$ is an integer
5. Linear Convolution in Time: $y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$
6. Circular Convolution in Time: $y[n] = h[n] \circledast x[n] \leftrightarrow Y[k] = H[k]X[k]$
7. Convolution in Frequenty: $y[n] = w[n]x[n] \leftrightarrow Y(\omega) = \frac{1}{2\pi} W(\omega) * X(\omega)$

Rectangular & Hamming Windows, Ideal LPF, and Tones

$$w_R[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = e^{-\frac{j\omega(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$w_H[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) w_R[n] \leftrightarrow W_H(\omega) = 0.54 W_R(\omega) - 0.23 W_R\left(\omega - \frac{2\pi}{N-1}\right) - 0.23 W_R\left(\omega + \frac{2\pi}{N-1}\right)$$

$$h_{\text{ideal}}[n] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \leftrightarrow H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$e^{j\omega_0 n} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

$$\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}} \leftrightarrow \frac{2\pi}{N} \sum_{k=0}^{N-1} X[k] \delta\left(\omega - \frac{2\pi kn}{N}\right)$$

Overlap-Add

$$N \ge M + L - 1$$
$$x_t[n] = x[n + tM] w[n]$$
$$Y_t[k] = X_t[k] H[k]$$
$$y[n] = \sum_t y_t[n - tM]$$

1. (9 points) Consider the FIR filter $h[n] = -\delta[n] + 2\delta[n-1] - \delta[n-2]$.

   (a) (6 points) Calculate the frequency response, $H(\omega)$, of this filter. Sketch $|H(\omega)|$ for $-\pi \leq \omega \leq \pi$.

   Solution:

   $$H(\omega) = -1 + 2e^{-j\omega} - e^{-2j\omega}$$
   $$|H(\omega)| = 2 - 2\cos(\omega)$$

   The sketch should show that $|H(0)| = 4$, $|H(\frac{\pi}{2})| = 2$, and $|H(\pi)| = 0$

   (b) (3 points) Is this a lowpass, highpass, bandpass, or bandstop filter? Why?

   Solution: Lowpass, because it emphasizes low frequencies while zeroing out high frequencies.

2. Use the windowing method to design a length-5 generalized linear phase FIR lowpass filter with a cutoff frequency of $\omega_c = \pi/4$.

   (a) (4 points) An ideal lowpass filter with cutoff $\omega_c = \frac{\pi}{4}$ has the following Fourier transform:

$$D(\omega) = \begin{cases} 1 & |\omega| < \frac{\pi}{4} \\ 0 & |\omega| > \frac{\pi}{4} \end{cases}$$

   Use the inverse Fourier transform formula to find $d[n]$, and sketch it as a function of n, for $-5 \leq n \leq 5$.

   Solution:

$$d[n] = \frac{1}{4}\text{sinc}\left(\frac{\pi n}{4}\right)$$

   The sketch should have a peak of $d[0] = 1/4$, and zero crossings whenever n is a multiple of 4.

   (b) (4 points) Choose any causal window of length $N = 5$ samples. Specify the name of the window you are using, and the mathematical formula used to calculate its coefficients. Sketch the window, $w[n]$, as a function of n, for $-5 \leq n \leq 5$.

   Solution: Answer could be rectangular, Hamming, Hann, or any other. For example,

$$w_{\text{Hamming}}[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{5}\right) & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

   Sketch should show a window of length 5 samples defined for $0 \leq n \leq 4$.

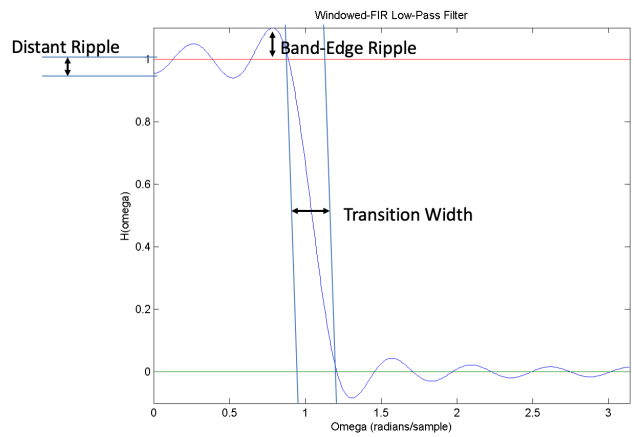
   (c) (4 points) Specify $h[n]$ in terms of $w[n]$ and $d[n]$ (that is, give an equation of the form "$h[n] = \ldots$" where "$\ldots$" is replaced by some function of $w[n]$ and $d[n]$). Sketch $h[n]$ as a function of n for $-5 \leq n \leq 5$.

   Solution:

$$h[n] = w[n]d[n-2]$$

   Sketch should show a filter with a peak of $h[2] = 1/4$, and with nonzero samples for $0 \leq n \leq 4$.

3. (20 points) An FIR filter will be created as $h[n] = w[n]d[n]$, for $D(\omega) = \begin{cases} 1 & |\omega| < \frac{\pi}{3} \\ 0 & \text{else} \end{cases}$. You are considering three possible windows: rectangular, Hamming, and Hann, each of length $2M + 1 = 23$ samples. These three windows are differentiated by three typical measurements, depicted in the figure below: transition width, band-edge ripple, and distant ripple.



Your evil cousin Morwen has calculated transition width, band-edge ripple, and distant ripple for the rectangular, Hamming, and Hann windows. Out of spite, however, she has mixed them up, so that you don't know which one goes with which window; all you know is that, for each of the three windows,

Transition Width	is either	$\frac{2\pi}{23}$ or $\frac{4\pi}{23}$
Band-Edge Ripple	is either	0.0033 or 0.0062 or 0.099
Distant Ripple	is either	0.0006 or 0.0031 or 0.045

Sort the values above into the table below.

Window	Rect	Hamm	Hann
Transition Width			
Band-Edge Ripple			
Distant Ripple			

Solution:

Window	Rect	Hamm	Hann
Transition	$\frac{2\pi}{23}$	$\frac{4\pi}{23}$	$\frac{4\pi}{23}$
Band-Edge R	0.099	0.0033	0.0062
Distant R	0.045	0.0031	0.0006

4. (20 points) Consider the signal $y[n]$ defined by

$$y[n] = \begin{cases} e^{j0.42\pi n} & |n| \leq 11 \\ 0 & \text{otherwise} \end{cases}$$

Use the windowing property ($w[n]x[n] \leftrightarrow \frac{1}{2\pi}W(\omega) * X(\omega)$) to find $Y(\omega)$, the DTFT of $y[n]$.

Solution:

$$Y(\omega) = \frac{\sin\left(\frac{23}{2}(\omega - 0.42\pi)\right)}{\sin\left(\frac{1}{2}(\omega - 0.42\pi)\right)}$$

5. (20 points) Find $y[n] = h[n] * x[n]$, where

$$x[n] = \cos(0.02\pi n), \quad h[n] = \begin{cases} 1 & |n| \leq 3 \\ 0 & |n| > 3 \end{cases}$$

What is $y[n]$?

Solution:

$$\begin{aligned} X(\omega) &= \pi\delta(\omega - 0.02\pi) + \pi\delta(\omega + 0.02\pi) \\ H(\omega) &= \frac{\sin(7\omega/2)}{\sin(\omega/2)} \\ Y(\omega) &= \frac{\pi \sin(0.07\pi)}{\sin(0.01\pi)} \left(\delta(\omega - 0.02\pi) + \delta(\omega + 0.02\pi)\right) \\ y[n] &= \frac{\sin(0.07\pi)}{\sin(0.01\pi)} \cos(0.02\pi n) \end{aligned}$$

6. (20 points) Suppose

$$x[n] = \cos\left(\frac{7\pi n}{21}\right), \quad y[n] = \begin{cases} x[n] & |n| \le 10 \\ 0 & \text{otherwise} \end{cases}, \quad Y(\omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n}$$

Sketch $Y(\omega)$ for $-\pi \le \omega \le \pi$. Specify the frequency and amplitude of at least one peak. Also, specify at least three particular frequencies ω such that $Y(\omega) = 0$.

Solution:

$$\begin{aligned} X(\omega) &= \pi\delta\left(\omega - \frac{7\pi}{21}\right) + \pi\delta\left(\omega + \frac{7\pi}{21}\right) \\ W(\omega) &= \frac{\sin(21\omega/2)}{\sin(\omega/2)} \\ Y(\omega) &= \frac{1}{2\pi} W(\omega) * X(\omega) \\ &= \frac{1}{2} W\left(\omega - \frac{7\pi}{21}\right) + \frac{1}{2} W\left(\omega + \frac{7\pi}{21}\right) \end{aligned}$$

The sketch should show two sinc functions with peaks at $\omega = \pm\frac{7\pi}{21}$ and with nulls at odd integer multiples of $\frac{\pi}{21}$.

7. (25 points) Consider the signal

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(a) Find the DTFT, $X(\omega)$.

Solution: $X(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

(b) Find the power spectrum $|X(\omega)|^2$, and sketch it for $-\pi \leq \omega \leq \pi$. Specify its values at $\omega = 0$, $\omega = \frac{\pi}{2}$, and $\omega = \pi$.

Solution:

$$|X(\omega)|^2 = \frac{1}{\frac{5}{4} - \cos\omega}$$

The sketch should show a smooth function of ω with a peak at $\omega = 0$ and with the values

$$|X(0)|^2 = 4$$
$$|X(\frac{\pi}{2})|^2 = \frac{4}{5}$$
$$|X(\pi)|^2 = \frac{4}{9}$$

8. (25 points) Suppose you want to design a band-stop filter as follows:

$$D(\omega) = \begin{cases} 1 & |\omega| < \frac{\pi}{5} \\ 0 & \frac{\pi}{5} < |\omega| < \frac{\pi}{2} \\ 1 & \text{otherwise} \end{cases}$$

(a) Find the desired impulse response, $d[n]$.

Solution:

$$d[n] = \delta[n]\left(\frac{1}{2}\right)\operatorname{sinc}\left(\frac{\pi n}{2}\right) + \left(\frac{1}{5}\right)\operatorname{sinc}\left(\frac{\pi n}{5}\right)$$

(b) Suppose you are willing to tolerate transition bands of up to $\Delta\omega = \frac{\pi}{32}$ radians/sample (measured from passband ripple to stopband ripple), but that you want the ripples to be as small as possible. What type of window should you use, and how long should it be?

Solution: Hamming window, with a length of $N = 256$ samples.

9. (10 points) Consider the following ideal bandpass filter:

$$H(\omega) = \begin{cases} 1 & \frac{\pi}{6} < |\omega| < \frac{\pi}{3} \\ 0 & \text{otherwise} \end{cases}$$

Find the impulse response, $h[n]$.

Solution:

$$h[n] = \frac{\sin(\pi n/3) - \sin(\pi n/6)}{\pi n}$$

10. (20 points) Consider the following filter:

$$h[n] = \begin{cases} 1 & -6 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Sketch $|H(\omega)|$ as a function of ω, for $-\pi < \omega < \pi$. Label the amplitude at $\omega = 0$, and at least two of the frequencies at which $|H(\omega)| = 0$.

Solution:

$$|H(\omega)| = \left| \frac{\sin(9\omega/2)}{\sin(\omega/2)} \right|$$

The sketch should show a sinc-like function of ω, with a peak at $\omega = 0$ of $|H(0)| = 9$. $|H(\omega)| = 0$ at the frequencies $\omega = \frac{2\pi k}{9}$, for any integer k that is not a multiple of 9.

11. (20 points) Consider the following filter:

$$h[n] = \begin{cases} 1 & -6 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Design a filter $g[n]$ such that $F(\omega)$ is real valued (i.e., its imaginary part is zero, $\Im\{F(\omega)\} = 0$), where

$$f[n] = g[n] * h[n]$$

$$g[n] =$$

Solution:

$$g[n] = \delta[n-2]$$

12. (20 points) The signal $x(t) = \cos(2\pi 800t)$ is sampled at $F_s = 8000$ samples/second, then processed by the filter

$$y[n] = \frac{1}{7}\sum_{m=-3}^{3} x[n-m]$$

The resulting signal is $y[n] = A\cos(\omega n + \theta)$. What are A, ω, and θ?

$A =$

Solution:

$$\begin{aligned} A &= \frac{\sin(2\pi 800 \times 7/(2 \times 8000))}{7\sin(2\pi 800/(2 \times 8000))} \\ &= \frac{\sin(7\pi/10)}{7\sin(\pi/10)} \end{aligned}$$

$\omega =$

Solution:

$$\omega = \frac{2\pi 800}{8000} = \frac{\pi}{5}$$

$\theta =$

Solution:

$$\theta = 0$$

13. (20 points) You want to approximate the following ideal bandpass filter:

$$H_{\text{ideal}}(\omega) = \begin{cases} 1 & 0.3\pi < |\omega| < 0.4\pi \\ 0 & \text{otherwise} \end{cases}$$

Design an FIR filter $h[n]$ that is exactly 64 samples long, such that $|H(\omega)| \approx |H_{\text{ideal}}(\omega)|$. Window $h[n]$ in such a way that the first sidelobe has a level of less than -40dB relative to the mainlobe.

$$h[n] =$$

Solution:

$$h[n] = w_H[n]\left(0.4\text{sinc}(0.4\pi(n-31.5)) - 0.3\text{sinc}(0.3\pi(n-31.5))\right)$$

where

$$w_H[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{63}\right) & 0 \le n \le 63 \\ 0 & \text{otherwise} \end{cases}$$

14. (16 points) It's hard to convolve two sinc functions, but it's easy to muliply their Fourier transforms. For example, suppose that

$$x[n] = \text{sinc}(0.5\pi n)$$
$$h[n] = \frac{1}{3}\text{sinc}\left(\frac{\pi n}{3}\right)$$
$$y[n] = x[n] * h[n]$$

Find $y[n]$ as a function of n; x and h should not appear in your answer.

Solution:

$$y[n] = \frac{2}{3}\text{sinc}\left(\frac{\pi n}{3}\right)$$

15. (17 points) Suppose $x[n] = \cos(0.32\pi n) w_H[n]$, where $w_H[n]$ is a nearly-Hamming window defined as follows:

$$w_H[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{100}\right) & 0 \le n \le 99 \\ 0 & \text{otherwise} \end{cases}$$

Let $X[k]$ be the 100-point DFT of $x[n]$. For which values of k, $0 \le k \le 99$, is $X[k]$ nonzero?

Solution:

$$\begin{aligned} X[k] = X(\omega) \text{ at } \omega &= \frac{2\pi k}{100} \\ &= W_H[k-16] + W_H[k+16] \\ &= W_H[k-16] + W_H[k-100+16] \end{aligned}$$

The normal $W_H[k]$ is nonzero everywhere, because the raised cosine has a frequency of $\frac{2\pi}{N-1}$. This modified Hamming window, however, has a frequency of $\frac{2\pi}{N}$, so we get that

$$W_H[k] = 0.54 W_R[k] + 0.23 W_R[k-1] + W_R[k+1]$$

Since $W_R[k]$ is nonzero only at $k = 0$, we have that $X[k]$ is nonzero only at

$$k \in \{15, 16, 17, 100-17, 100-16, 100-15\}$$

16. (17 points) Suppose $h[n]$ is a periodically repeated, rectangular-windowed ideal lowpass filter of length 94 samples

$$h[n] = \begin{cases} \frac{1}{3}\text{sinc}\left(\frac{\pi n}{3}\right) & 0 \leq n \leq 46 \\ \frac{1}{3}\text{sinc}\left(\frac{\pi}{3}(n-94)\right) & 47 \leq n \leq 93 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $x[n] = \delta[n-15]$, $X[k]$ is the DFT of $x[n]$, and $Y[k] = H[k]X[k]$. Find $y[n]$, the inverse DFT of $Y[k]$.

Solution:

$$y[n] = \begin{cases} \frac{1}{3}\text{sinc}\left(\frac{\pi(n-15)}{3}\right) & 0 \leq n \leq 61 \\ \frac{1}{3}\text{sinc}\left(\frac{\pi}{3}(n-109)\right) & 62 \leq n \leq 93 \\ 0 & \text{otherwise} \end{cases}$$

17. (25 points) You have an application for which it is necessary to increase the amplitude of inputs below $\frac{\pi}{4}$ radians/sample, while eliminating inputs above $\frac{3\pi}{4}$ radians/sample. The ideal filter is therefore:

$$H_i(\omega) = \begin{cases} 2 & |\omega| < \frac{\pi}{4} \\ 1 & \frac{\pi}{4} < |\omega| < \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} < |\omega| < \pi \end{cases}$$

(a) What is $h_i[n]$?

Solution: There are several ways to write it. One is:

$$h_i[n] = \frac{3}{4}\text{sinc}\left(\frac{3\pi n}{4}\right) + \frac{1}{4}\text{sinc}\left(\frac{\pi n}{4}\right)$$

(b) You propose to create a realizable filter, $h[n]$, by windowing $h_i[n]$ with a length-127 Hamming window: $h[n] = w[n]h_i[n]$. In the resulting frequency response, $H(\omega)$, how wide are the transition bands?

Solution: $\frac{4\pi}{127}$ radians/sample.

18. (25 points) A linear shift-invariant system has the following frequency response:

$$H(\omega) = e^{-5j\omega}\left(1 + \frac{1}{2}\cos\omega\right)$$

(a) Suppose that $y[n] = h[n] * x[n]$, where

$$x[n] = 3\sin\left(\frac{\pi n}{6}\right)$$

What is $y[n]$?

Solution:

$$y[n] = 3\left(1 + \frac{1}{2}\cos(\pi/6)\right)\sin\left(\frac{\pi(n-5)}{6}\right)$$

(b) What is the impulse response, $h[n]$, of this system?

Solution:

$$h[n] = \frac{1}{4}\delta[n-4] + \delta[n-5] + \frac{1}{4}\delta[n-6]$$

19. (20 points) Suppose you want an FIR bandpass filter with a length of $N = 1024$, with cutoff frequencies $\omega_{\text{LO}} = 0.46\pi$ and $\omega_{\text{HI}} = 0.48\pi$, and with little stopband ripple. Find an impulse response $h[n]$ that meets these requirements.

Solution: The solution is the difference of two ideal lowpass filters, shifted by an odd number of half-samples (e.g., $\frac{1023}{2}$), then windowed so that it is symmetric and has 1024 samples (e.g., $0 \leq n \leq 1023$). To minimize stop-band ripple, it should be windowed by some tapered window, thus:

$$h[n] = w[n]\left(0.48\text{sinc}\left(0.48\pi\left(n - \frac{1023}{2}\right)\right) - 0.46\text{sinc}\left(0.46\pi\left(n - \frac{1023}{2}\right)\right)\right),$$

where $w[n]$ is a tapered window such as a Hamming, Hann, or Bartlett window, e.g.,

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{1023}\right) & 0 \leq n \leq 1023 \\ 0 & \text{otherwise} \end{cases}$$

20. (15 points) Consider the signal

$$x[n] = \cos\left(\frac{\pi n}{100}\right)$$

What is the 128-sample DFT, $X[k]$, of this signal? Be sure to consider windowing effects.

Solution:

$$w[n]x[n] = \frac{1}{2}w[n]e^{j\pi n/100} + \frac{1}{2}w[n]e^{-j\pi n/100}$$

$$\text{DTFT}\left\{w[n]x[n]\right\} = \frac{1}{2}W\left(\omega - \frac{\pi}{100}\right) + \frac{1}{2}W\left(\omega + \frac{\pi}{100}\right),$$

where $w[n]$ is a rectangular window, so

$$W(\omega) = e^{-j\omega\frac{127}{2}}\frac{\sin(64\omega)}{\sin(\omega/2)}$$

$$\begin{aligned} X[k] &= \text{DTFT}\left\{w[n]x[n]\right\}\Big|_{\omega=\frac{2\pi k}{128}} \\ &= \frac{1}{2}W\left(\frac{2\pi k}{128} - \frac{\pi}{100}\right) + \frac{1}{2}W\left(\frac{2\pi k}{128} + \frac{\pi}{100}\right) \end{aligned}$$

21. (25 points) Suppose you have a very long signal, $x[n]$, that you want to filter to compute $y[n] = h[n]*x[n]$. This can be done using the following sequence of steps:

$$H[k] = \sum_{n=0}^{N-1} h[n]e^{-j\frac{2\pi kn}{N}} \tag{1}$$

$$x_\ell[n] = \begin{cases} x[n+\ell M] & 0 \le n \le M-1, \text{ for all } \ell \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$X_\ell[k] = \sum_{n=0}^{N-1} x_\ell[n]e^{-j\frac{2\pi kn}{N}} \tag{3}$$

$$Y_\ell[k] = H[k]X_\ell[k] \tag{4}$$

$$y_\ell[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y_\ell[k]e^{j\frac{2\pi kn}{N}} \tag{5}$$

$$y[n] = \sum_{\ell=-\infty}^{\infty} y_\ell[n-\ell M] \tag{6}$$

(a) Suppose $h[n]$ is 129 samples long. Find values of M and N such that the algorithm in Eqs. (1) through (6) gives the same result as $y[n] = h[n] * x[n]$. There are many different correct answers; you only need to find one correct answer.

Solution: Any pair of values such that $N \ge L + M - 1$ is a valid answer. For example, you could say that $M = 128$ and $N = 256$.

(b) Suppose $x[n] = \cos(0.08\pi n)$, and $h[n] = \delta[n-126]$. Find $x_\ell[n]$ and $y_\ell[n]$ in terms of ℓ, M, and n.

Solution:

$$x_\ell[n] = \begin{cases} \cos(0.08\pi(n+\ell M)) & 0 \le n \le M-1, \text{ for all } \ell \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} y_\ell[n] &= h[n] \circledast x_\ell[n] \\ &= x_\ell\left[\langle n-126\rangle_N\right] \\ &= \begin{cases} \cos(0.08\pi(n+\ell M-126)) & 126 \le n \le M+125, \text{ for all } \ell \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

22. (30 points) This problem is about the design of a lowpass filter.

(a) The desired frequency response is

$$F(\omega) = \begin{cases} 1 & |\omega| \leq \frac{\pi}{5} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

What is $f[n]$?

Solution:

$$f[n] = \left(\frac{1}{5}\right) \operatorname{sinc}\left(\frac{\pi n}{5}\right)$$

(b) As a first step toward implementation, the filter is truncated, thus

$$g[n] = \begin{cases} f[n] & |n| \leq 64 \\ 0 & \text{otherwise} \end{cases}$$

The resulting frequency response can be writtten as

$$G(\omega) = \frac{1}{2\pi} F(\omega) * W(\omega), \tag{8}$$

where $F(\omega)$ is the function shown in Eq. (7), and $W(\omega)$ is some function of ω. Find $W(\omega)$ as a function of ω.

Solution:

$$W(\omega) = \frac{\sin(129\omega/2)}{\sin(\omega/2)}$$

(c) As a second step toward implementation, the filter is delayed by 64 samples, thus

$$h[n] = g[n - 64]$$

The resulting frequency response can be written as $H(\omega) = D(\omega)G(\omega)$, where $G(\omega)$ is the function shown in Eq. (8), and $D(\omega)$ is some function of ω. Find $D(\omega)$ as a function of ω.

Solution:

$$D(\omega) = e^{-j64\omega}$$

23. (24 points) Suppose that the frequency response of a system is:

$$H(\omega) = \begin{cases} 67 & -\frac{\pi}{5} \leq \omega \leq \frac{2\pi}{5} \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $h[n]$, the impulse response of this system, for all n. Notice that $h[n]$ is complex-valued.

Solution: All of the following are acceptable solutions:

$$\begin{aligned} h[n] &= \frac{67}{2\pi jn}\left(e^{jn2\pi/5} - e^{-jn\pi/5}\right) \\ &= \frac{67}{\pi n} e^{jn\pi/10} \sin(3\pi n/10) \\ &= 67 \times \frac{3}{10} e^{jn\pi/10} \text{sinc}(3\pi n/10) \end{aligned}$$

24. (24 points) Consider the following signal:

$$w[n] = \begin{cases} 0.1 & -27 \leq n < -12 \\ 0 & \text{otherwise} \end{cases}$$

Note that the DTFT of this signal can be expressed as $W(\omega) = e^{j\omega b} R(w)$ for some real-valued scalar b and real-valued function $R(\omega)$. Find b and $R(\omega)$.

Solution:

$$\begin{aligned} b &= 20 \\ R(\omega) &= 0.1 \frac{\sin(15\omega/2)}{\sin(\omega/2)} \end{aligned}$$

25. (22 points) Suppose that

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$
$$z[n] = y[n] - y[n-10]$$
$$H(\omega) = \frac{Z(\omega)}{X(\omega)}$$

Find $H(\omega)$.

Solution:

$$h[n] = u[n] - u[n-10]$$

Therefore

$$H(\omega) = e^{-j\frac{9\omega}{2}} \frac{\sin(5\omega)}{\sin(\omega/2)}$$