

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 SIGNAL PROCESSING
PRACTICE EXAM 1

Exam will be October 5, 2026, in class

- This practice exam is about 9 times the expected length of the actual exam.
- This is a **CLOSED BOOK** exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not leave any unresolved integrals or summations in your answer. Other than that, however, you should not simplify explicit numerical expressions.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must **SHOW YOUR WORK** to get full credit.

NetID: _____

Linearity, Shift-Invariance, Causality, Stability

1. **Linearity:** $x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$
2. **Shift-Invariance:** $x_1[n - n_0] \rightarrow y_1[n - n_0]$
3. **Causality:** $h[n] = 0$ for $n < 0$
4. **Stability:** $|x[n]| < M_1 \Rightarrow |y[n]| < M_2$

Convolution

$$h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Sampling and Interpolation:

$$\begin{aligned}x[n] &= x\left(t = \frac{n}{F_s}\right) \\f_a &= \min(f \bmod F_s, -f \bmod F_s) \\z_a &= \begin{cases} z & f \bmod F_s < -f \bmod F_s \\ z^* & f \bmod F_s > -f \bmod F_s \end{cases} \\y(t) &= \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)\end{aligned}$$

Fourier Series

$$\begin{aligned}\text{Analysis: } X_k &= \frac{1}{T_0} \int_0^{T_0} x(t)e^{-j2\pi kt/T_0} dt \\ \text{Synthesis: } x(t) &= \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}\end{aligned}$$

Discrete Fourier Transform

$$\begin{aligned}\text{Analysis: } X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \\ \text{Synthesis: } x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \\ \text{Circular Convolution: } x[n] \circledast h[n] &= \sum_{m=0}^{N-1} h[m]x[(n-m)_N]\end{aligned}$$

Frequency Response

$$\begin{aligned}H(\omega) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ h[n] * \cos(\omega n) &= |H(\omega)| \cos(\omega n + \angle H(\omega))\end{aligned}$$

1. (25 points) Consider the following periodic signal:

$$x(t) = \begin{cases} 5e^{-400t} & 0 \leq t < 0.01 \\ x(t - 0.01) & \text{otherwise} \end{cases}$$

Find the Fourier series coefficients of this signal, X_k , including X_0 . The only variable appearing in your answer should be k , but your answer may also include numerical constants such as e , π , and j .

2. (25 points) Consider the signal

$$x_a(t) = \cos\left(2\pi 3300t + \frac{\pi}{6}\right) + \cos\left(2\pi 7300t + \frac{\pi}{4}\right) + \cos\left(2\pi 8300t + \frac{\pi}{3}\right)$$

Suppose that this signal is sampled at 8000 samples/second, to create the signal $x[n]$:

$$x[n] = \cos(\omega_1 n + \theta_1) + \cos(\omega_2 n + \theta_2) + \cos(\omega_3 n + \theta_3)$$

Find values of ω_k and θ_k , $k \in \{1, 2, 3\}$, such that $0 \leq \omega_k \leq \pi$ and $x[n] = x_a(n/8000)$.

3. (25 points) Consider the following discrete-time signal, $x[n]$:

$$x[n] = \sin\left(\frac{2\pi n}{8}\right),$$

thus $x[n]$ is periodic with a period of 8, and its first 8 samples are $\left\{0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}\right\}$. Suppose that $x_a(t) = \sum_{n=-\infty}^{\infty} x[n]p(t - nT)$, where $T = 0.001$ seconds, and $p(t)$ is a triangular interpolation kernel:

$$p(t) = \max(1 - 1000|t|, 0)$$

- (a) (15 points) Sketch the period of $x_a(t)$ for $0 \leq t \leq 0.008$, or else specify its values using an equation.

- (b) (10 points) What frequencies are present in the spectrum of $x_a(t)$?

4. (25 points) Consider a system whose output, $y[n]$, is an average of the samples of its input:

$$y[n] = \sum_{m=m_1}^{m_2} x[n-m]$$

The behavior of the system depends on the values of m_1 and m_2 . You may assume that $m_1 < m_2$, but you should consider all possibilities in the range $-\infty \leq m_1 < m_2 \leq \infty$.

- (a) Are there any values of m_1 or m_2 for which this system is nonlinear? If so, list all values of m_1 and m_2 for which the system is nonlinear. If not, prove that the system is linear.

- (b) Are there any values of m_1 or m_2 for which this system is shift-varying? If so, list all values of m_1 and m_2 for which the system is shift-varying. If not, prove that the system is shift-invariant.

- (c) Are there any values of m_1 or m_2 for which this system is non-causal? If so, list all values of m_1 and m_2 for which the system is non-causal. If not, prove that the system is causal.

- (d) Are there any values of m_1 or m_2 for which this system is unstable? If so, list all values of m_1 and m_2 for which the system is unstable. If not, prove that the system is stable.

5. (25 points) Suppose the input to a system is $x[n] = u[n] - u[n - L]$, and the impulse response is $h[n] = a^n u[n]$, where $u[n]$ is the unit step function ($u[n] = 1$ for $n \geq 0$, $u[n] = 0$ for $n < 0$). Find $x[n] * h[n]$ for all n as a function of a and L .

6. (25 points) In a switching power supply, power is delivered to a circuit in the form of a periodic square wave, $x(t)$, with a period of T_0 seconds, and with an adjustable parameter R ($0 < R < 1$) that is called the “duty cycle:”

$$x(t) = \begin{cases} 1 & 0 < t < RT_0 \\ 0 & RT_0 < t < T_0 \end{cases}$$

In terms of R , what are the Fourier series coefficients X_k ? Calculate X_k for all values of k , including $k = 0$. There should be no unresolved integrals in your answer, and you should find that it is possible to express the answer without the variables t, T_0 , or F_0 ; other simplifications are not necessary.

7. (25 points) The voltage across a cardioid microphone is

$$m(t) = ap(t) + (1 - a)\frac{dp}{dt},$$

where $p(t)$ is the air pressure, as a function of time, at the microphone's location, and a is a coefficient, $0 < a < 1$, that depends on the direction from which the wave arrives. The microphone voltage is usually enhanced by a pre-amplifier with a gain of G , then transmitted on a wire until it reaches the A/D converter; the voltage at input to the A/D converter is delayed by a short delay of τ seconds, so

$$v(t) = Gm(t - \tau)$$

Suppose that the signal being recorded is the voice of a tenor, singing the A above middle C, thus

$$\begin{aligned} p(t) &= \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t} \\ m(t) &= \sum_{k=-\infty}^{\infty} M_k e^{j2\pi k F_0 t} \\ v(t) &= \sum_{k=-\infty}^{\infty} V_k e^{j2\pi k F_0 t} \end{aligned}$$

Express V_k in terms of X_k , F_0 , k , a , G , τ , and k .

8. (25 points) A signal $x(t)$ is sampled at 16,000 samples/second, then played back through an ideal D/A, thus:

$$x(t) = 14 \cos \left(2\pi 6000t + \frac{3\pi}{4} \right) + 2 \cos \left(2\pi 12,000t - \frac{\pi}{4} \right)$$

$$x[n] = x \left(t = \frac{n}{16,000} \right)$$

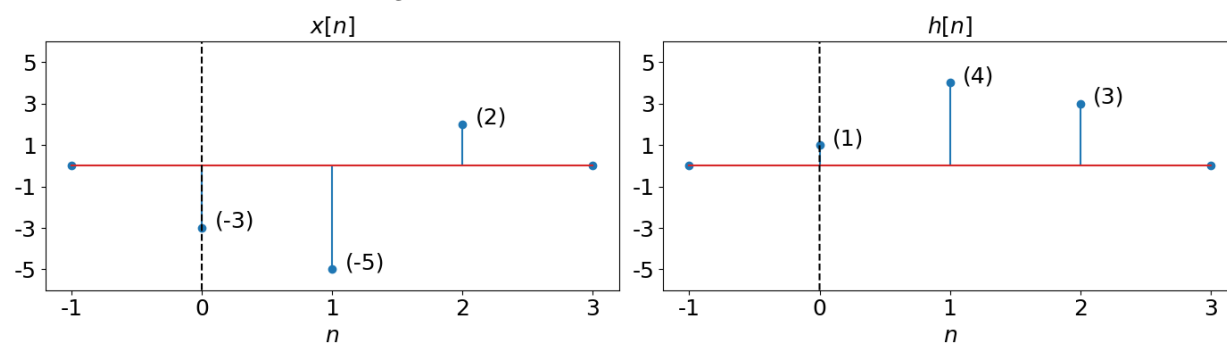
$$y(t) = \sum_{n=-\infty}^{\infty} x[n] \text{sinc}(16,000\pi t - \pi n)$$

where the sinc function is defined as

$$\text{sinc}(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin(x)}{x} & \text{otherwise} \end{cases}$$

Find $y(t)$.

9. (25 points) The following figure shows two signals, $x[n]$ and $h[n]$. Each signal has three nonzero samples, with values as shown in the figure.



Find $y[n] = h[n] * x[n]$. Plot your answer using a stem plot similar to the plot shown above, and label the value of each nonzero sample in parentheses next to the sample, as shown above.

10. (25 points) Frequency modulation (FM) is a system that accepts an input signal $x[n]$, and computes an output signal $y[n]$ according to

$$y[n] = e^{j(\omega_0 + x[n])n},$$

where ω_0 is a real scalar constant called the carrier frequency.

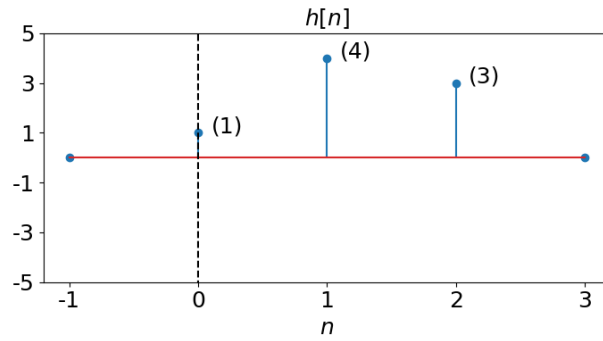
(a) Is FM linear? Prove your answer.

(b) Is FM shift-invariant? Prove your answer.

(c) Is FM causal? Prove your answer.

(d) Is FM stable? Prove your answer.

11. (25 points) Suppose the signal $x[n] = e^{j\omega_0 n}$ is passed through a linear shift-invariant system with the following impulse response:



Find the system output. Your answer should be a function of only n and ω_0 .

12. (25 points) The system $\mathcal{R}$ computes the ratio of two consecutive samples:

$$\mathcal{R} : x[n] \rightarrow y[n] = \frac{x[n]}{x[n-1]}$$

(a) Is $\mathcal{R}$ linear? Prove your answer.

(b) If $\mathcal{R}$ time-invariant? Prove your answer.

13. (25 points) Consider the following signal:

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 45 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X[k]$ is the length-64 DFT of $x[n]$. Find $X[k]$ as a function of k .

14. (22 points) Consider the following signal:

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 45 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X[k]$ is the length-64 DFT of $x[n]$, $Y[k] = X^2[k]$, and $y[n]$ is the 64-point inverse DFT of $Y[k]$. Find $y[n]$ as a function of n .

15. (25 points) $x(t)$ is a signal with a period of 0.01 seconds, and with the following shape:

$$x(t) = \begin{cases} 1 & 0 < t < 0.001 \\ 0 & 0.001 < t < 0.005 \\ -1 & 0.005 < t < 0.006 \\ 0 & 0.006 < t < 0.01 \end{cases}$$

(a) What are the Fourier series coefficients X_k for $k \neq 0$? Your answer should contain no variables other than k , but you don't need to simplify.

(b) Suppose that $y(t)$ is a signal such that $x(t) = \frac{dy}{dt}$. Express the Fourier series coefficients Y_k in terms of the Fourier series coefficients X_k . Note that you don't need to solve part (a) in order to solve this part of the problem.

16. (25 points) Suppose $x(t)$ is sampled to create the signal $y[n] = x\left(\frac{n}{F_s}\right)$, with a sampling frequency of $F_s = 10,000$ samples/second. The signal $y[n]$ is then passed through an ideal D/A in order to produce the signal $z(t)$. In each of the two following cases, what is $z(t)$?

(a)

$$x(t) = 3 \cos\left(2\pi 8000t + \frac{\pi}{4}\right)$$

What is $z(t)$?

(b)

$$x(t) = 3 \cos\left(2\pi 12,000t + \frac{\pi}{4}\right)$$

What is $z(t)$?

17. (25 points) Suppose that

$$x[n] = \sin\left(\frac{\pi n}{2}\right) = \begin{cases} 1 & n \text{ is odd, and } \frac{n-1}{2} \text{ is even} \\ -1 & n \text{ is odd, and } \frac{n-1}{2} \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

You would like to generate a continuous-time audio signal, $y(t)$, using the interpolation formula

$$y(t) = \sum_{n=-\infty}^{\infty} x[n]g(t-n)$$

In each of the following cases, specify the value of $y(t)$ over the range $0 \leq t \leq 4$. You may specify $y(t)$ by drawing a plot of the function (if your plot clearly shows the value at each point in time in the range $0 \leq t \leq 4$), or by using an equation or a set of cases.

(a) What is $y(t)$ if

$$g(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

(b) What is $y(t)$ if

$$g(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) What is $y(t)$ if

$$g(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & \text{otherwise} \end{cases}$$

18. (25 points) Consider the following system:

$$y[n] = \begin{cases} x[n] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

(a) Is this system linear? Prove your answer.

(b) Is this system shift-invariant? Prove your answer.

19. (25 points) The second-derivative of a sampled signal may be approximated by an LSI system with the following impulse response:

$$h[n] = -\delta[n] + \frac{1}{2} (\delta[n+1] + \delta[n-1])$$

(a) Is this system causal? Why or why not?

(b) Is this system stable? Why or why not?

(c) What is $h[n] * h[n]$? Give the time and value of every nonzero sample, using either a sketch or an equation.

20. (25 points) Consider an LSI system with the following impulse response:

$$h[n] = -\delta[n] + \frac{1}{2} (\delta[n+1] + \delta[n-1])$$

(a) What is its frequency response?

(b) Consider a system $\mathcal{G}$ that accepts as input a signal $x[n]$, and generates as output a signal $z[n]$. The system $\mathcal{G}$ does two things to $x[n]$. First, it convolves it with $h[n]$, producing $y[n] = x[n] * h[n]$. Second, it delays it by five samples, producing $z[n] = y[n-5]$. What is the impulse response of the system $\mathcal{G}$?

21. (25 points) Suppose we have a continuous-time signal $x(t)$, given by

$$x(t) = (3 + 2j)e^{-j12\pi t/T_0} + (5 - j)e^{-j6\pi t/T_0} + (5 + j)e^{j6\pi t/T_0} + (3 - 2j)e^{j12\pi t/T_0}$$

- (a) Let's multiply $x(t)$ by the cosine of $4\pi t/T_0$, and integrate over one period:

$$A = \int_0^{T_0} x(t) \cos\left(\frac{4\pi t}{T_0}\right) dt$$

What is A ?

- (b) Suppose we multiply, instead, by the cosine of $6\pi t/T_0$, and integrate over one period:

$$B = \int_0^{T_0} x(t) \cos\left(\frac{6\pi t}{T_0}\right) dt$$

What is B ?

- (c) Continuing with the same $x(t)$: suppose we sample $x(t)$ with a sampling period of $T = T_0/8$, thus

$$y[n] = x(t)|_{t=nT}$$

The resulting $y[n]$ is periodic with a period of 8 samples, and with a discrete-time Fourier series of

$$y[n] = \sum_{k=-3}^3 Y_k e^{j\frac{2\pi kn}{8}}$$

Find the values of Y_k , for $-3 \leq k \leq 3$.

22. (20 points) Consider a linear shift-invariant system with the following impulse response:

$$h[n] = \begin{cases} (0.9)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(a) Is this system stable? Why or why not?

(b) Suppose that $y[n] = h[n] * x[n]$, where

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

Use convolution to find $y[n]$. You may find it useful to know that $\sum_{n=0}^{L-1} a^n = \frac{1-a^L}{1-a}$.

23. (25 points) A linear shift-invariant system has the following frequency response:

$$H(\omega) = e^{-5j\omega} \left(1 + \frac{1}{2} \cos \omega \right)$$

(a) Suppose that $y[n] = h[n] * x[n]$, where

$$x[n] = 3 \sin \left(\frac{\pi n}{6} \right)$$

What is $y[n]$?

(b) What is the impulse response, $h[n]$, of this system?

24. (25 points) Consider the signal

$$x[n] = \cos\left(\frac{\pi n}{100}\right)$$

(a) What is the 128-sample DFT, $X[k]$, of this signal?

(b) Suppose you want to construct the following periodic signal:

$$y[n] = \cos\left(\frac{\pi(n - 128\ell)}{100}\right), \quad 128\ell \leq n < 128(\ell + 1), \quad \text{for all } \ell$$

You can create this signal using the following Fourier series:

$$y[n] = \sum_{k=0}^{127} Y_k e^{j \frac{2\pi k n}{128}}$$

Notice that there is a relationship between Y_k and the DFT coefficients, $X[k]$, that you computed in part (a) of this problem. Find Y_k in terms of $X[k]$.

25. (25 points) $x(t)$ is a triangle wave with fundamental frequency of $F_0 = 100\text{Hz}$, and Fourier series coefficients of

$$X_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{j2\pi k^2} (-1)^{\frac{|k|-1}{2}} & k \text{ odd} \\ 0 & k \neq 0, k \text{ even} \end{cases}$$

The signal $y(t)$ is created by delaying $x(t)$ by 0.001 seconds, then differentiating it, i.e.,

$$y(t) = \frac{dx(t - 0.001)}{dt}$$

What are the Fourier series coefficients Y_k ?

26. (25 points) $x(t)$ is a signal with a period of 4 seconds, and with the following shape:

$$x(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \\ -2 & 2 < t < 3 \\ 0 & 3 < t < 4 \end{cases}$$

(a) What is F_0 ?

(b) What is X_0 , the 0th Fourier coefficient?

(c) What are the Fourier series coefficients X_k for $k \neq 0$?

27. (25 points) Suppose $x(t)$ is sampled to create the signal $y[n] = x\left(\frac{n}{F_s}\right)$, with a sampling frequency of $F_s = 8000$ samples/second. The signal $y[n]$ is then passed through an ideal D/A in order to produce the signal $z(t)$. In each of the two following cases, what is $z(t)$?

(a)

$$x(t) = 3 \cos(2\pi 8000t)$$

(b)

$$x(t) = 2 \cos\left(2\pi 4800t + \frac{\pi}{4}\right)$$

28. (25 points) Suppose that $x[n]$ is an audio signal sampled at $F_s = 10,000$ samples/second. You would like to generate a continuous-time audio signal, $y(t)$, using the interpolation formula

$$y(t) = \sum_{n=-\infty}^{\infty} x[n]g\left(t - \frac{n}{10000}\right)$$

You are considering several different levels of quality. In each of the following cases, write an equation for $g(t)$, or sketch it as a function of t . If you sketch $g(t)$, you must label its amplitude, and at least two of its zero-crossings.

- (a) Suppose you want $y(t)$ to be a piece-wise constant interpolation of $x[n]$. What value of $g(t)$ would make this true?

- (b) Suppose you want $y(t)$ to be a piece-wise linear interpolation of $x[n]$. What value of $g(t)$ would make this true?

- (c) Suppose you want $y(t)$ to be a perfectly bandlimited reconstruction of $x[n]$, with energy only in the frequency band $-5000 \leq f \leq 5000$. What value of $g(t)$ would make this true?

29. (20 points) Consider the following filter:

$$h[n] = \begin{cases} 1 & -6 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Sketch $|H(\omega)|$ as a function of ω , for $-\pi < \omega < \pi$. Label the amplitude at $\omega = 0$, and at least two of the frequencies at which $|H(\omega)| = 0$.

30. (25 points) The signal $x(t) = \cos(2\pi 800t)$ is sampled at $F_s = 8000$ samples/second, then processed by the filter

$$y[n] = \frac{1}{7} \sum_{m=-3}^3 x[n-m]$$

The resulting signal is $y[n] = A \cos(\omega n + \theta)$. What are A , ω , and θ ?

$$A =$$

$$\omega =$$

$$\theta =$$

31. (25 points) Consider the system

$$y[n] = \frac{1}{4}x[n+2] + \frac{1}{2}x[n+1] + x[n] - x[n-1] - \frac{1}{2}x[n-2] - \frac{1}{4}x[n-3]$$

Sketch the impulse response of this system as a function of n , for $-5 \leq n \leq 5$. Clearly label the values of every nonzero sample.

32. (25 points) $x(t)$ and $y(t)$ are two signals that are both periodic with fundamental frequency F_0 , i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}, \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j2\pi k F_0 t}$$

The relationship between $x(t)$ and $y(t)$ is given by

$$y(t) = x(t - 0.001) + 3 \frac{dx}{dt}$$

Find Y_k in terms of X_k and F_0 .

33. (25 points) A zebra is photographed at a resolution such that each black stripe is 3.5 pixels wide, and each white strip is 3.5 pixels wide. Modeling this as a square wave, we get the following model:

$$x[n] = 128 + 81 \cos\left(\frac{\pi n}{7}\right) + 27 \cos\left(\frac{3\pi n}{7}\right)$$

Unfortunately, the lens has some blur, so the image that is actually recorded is

$$y[n] = \frac{3}{4}x[n] + \frac{1}{4}x[n-1] \tag{1}$$

For some appropriate values of A_1, θ_1, A_3 , and θ_3 , $y[n]$ is equal to

$$y[n] = 128 + A_1 \cos\left(\frac{\pi n}{7} + \theta_1\right) + A_3 \cos\left(\frac{3\pi n}{7} + \theta_3\right) \tag{2}$$

Find explicit numerical expressions for A_1, θ_1, A_3 , and θ_3 such that Eq. (2) is equal to Eq. (1).

34. (25 points) Consider the signal

$$x[n] = 3\delta[n] + 2\delta[n - 1] + \delta[n - 2]$$

Find $X[k]$, the 4-point DFT of $x[n]$. The variable k should not appear in your answer; instead, please write explicit numerical expressions for the four samples of $X[k]$, $k \in \{0, 1, 2, 3\}$.

35. (25 points) Suppose

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 15 \text{ or } 49 \leq n \leq 63 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X[k]$ is the 64-point DFT of $x[n]$, and

$$Y[k] = e^{-j\pi k} X[k]$$

Find $y[n]$, the inverse DTFT of $Y[k]$.

36. (25 points) Suppose $h[n]$ is a length-94 signal given by:

$$h[n] = \begin{cases} \frac{1}{3} \text{sinc}\left(\frac{\pi n}{3}\right) & 0 \leq n \leq 46 \\ \frac{1}{3} \text{sinc}\left(\frac{\pi}{3}(n - 94)\right) & 47 \leq n \leq 93 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $x[n] = \delta[n - 15]$, $X[k]$ is the 94-sample DFT of $x[n]$, and $Y[k] = H[k]X[k]$. Find $y[n]$, the 94-sample inverse DFT of $Y[k]$.