Lecture 7: Sampling and Aliasing

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ECE 401: Signal and Image Analysis, Fall 2022
1. Review: Spectrum of continuous-time signals
2. Sampling
3. Aliasing
4. Aliased Frequency
5. Aliased Phase
6. Summary
7. Written Example
Outline

1. Review: Spectrum of continuous-time signals
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3. Aliasing
4. Aliased Frequency
5. Aliased Phase
6. Summary
7. Written Example
The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum } (x(t)) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$
Fourier’s theorem

One reason the spectrum is useful is that any periodic signal can be written as a sum of cosines. Fourier’s theorem says that any \(x(t)\) that is periodic, i.e.,

\[x(t + T_0) = x(t)\]

can be written as

\[x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kF_0 t}\]

which is a special case of the spectrum for periodic signals: \(f_k = kF_0\), and \(a_k = X_k\), and

\[F_0 = \frac{1}{T_0}\]
Fourier Series

- **Analysis** (finding the spectrum, given the waveform):

  \[ X_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0} dt \]

- **Synthesis** (finding the waveform, given the spectrum):

  \[ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} \]
Outline

1. Review: Spectrum of continuous-time signals
2. Sampling
3. Aliasing
4. Aliased Frequency
5. Aliased Phase
6. Summary
7. Written Example
How to sample a continuous-time signal

Suppose you have some continuous-time signal, \( x(t) \), and you’d like to sample it, in order to store the sample values in a computer. The samples are collected once every \( T_s = \frac{1}{F_s} \) seconds:

\[
x[n] = x(t = nT_s)
\]
Example: a 1kHz sine wave

For example, suppose \( x(t) = \sin(2\pi 1000t) \). By sampling at \( F_s = 16000 \) samples/second, we get

\[
x[n] = \sin \left( 2\pi 1000 \frac{n}{16000} \right) = \sin(\pi n/8)
\]
Outline

1. Review: Spectrum of continuous-time signals
2. Sampling
3. Aliasing
4. Aliased Frequency
5. Aliased Phase
6. Summary
7. Written Example
Can every sine wave be reconstructed from its samples?

The question immediately arises: can every sine wave be reconstructed from its samples? The answer, unfortunately, is “no.”
Can every sine wave be reconstructed from its samples?

For example, two signals $x_1(t)$ and $x_2(t)$, at 10kHz and 6kHz respectively:

$$x_1(t) = \cos(2\pi 10000t), \quad x_2(t) = \cos(2\pi 6000t)$$

Let's sample them at $F_s = 16,000$ samples/second:

$$x_1[n] = \cos \left( 2\pi 10000 \frac{n}{16000} \right), \quad x_2[n] = \cos \left( 2\pi 6000 \frac{n}{16000} \right)$$

Simplifying a bit, we discover that $x_1[n] = x_2[n]$. We say that the 10kHz tone has been “aliased” to 6kHz:

$$x_1[n] = \cos \left( \frac{5\pi n}{4} \right) = \cos \left( \frac{3\pi n}{4} \right)$$

$$x_2[n] = \cos \left( \frac{3\pi n}{4} \right) = \cos \left( \frac{5\pi n}{4} \right)$$
Can every sine wave be reconstructed from its samples?
What is the highest frequency that can be reconstructed?

The highest frequency whose cosine can be exactly reconstructed from its samples is called the “Nyquist frequency,” $F_N = F_S/2$. If $x(t) = \cos(2\pi F_N t)$, then

$$x[n] = \cos \left( 2\pi F_N \frac{n}{F_S} \right) = \cos(\pi n) = (-1)^n$$
Sampling above Nyquist ⇒ Aliasing to a frequency below Nyquist

If you try to sample a signal whose frequency is above Nyquist (like the one shown on the left), then it gets aliased to a frequency below Nyquist (like the one shown on the right).
Outline

1. Review: Spectrum of continuous-time signals
2. Sampling
3. Aliasing
4. Aliased Frequency
5. Aliased Phase
6. Summary
7. Written Example
Suppose you have a cosine at frequency $f$: 

$$x(t) = \cos(2\pi ft)$$

Suppose you sample it at $F_s$ samples/second. If $F_s$ is not high enough, it might get aliased to some other frequency, $f_a$. 

$$x[n] = \cos\left(2\pi fn/F_s\right) = \cos\left(2\pi f_a n/F_s\right)$$

How can you predict what $f_a$ will be?
Aliasing comes from two sources:

\[
\cos(\phi) = \cos(2\pi n - \phi)
\]

\[
\cos(\phi) = \cos(\phi - 2\pi n)
\]

The equations above are true for any integer \( n \).
Aliased Frequency

Let’s plug in $\phi = \frac{2\pi fn}{F_s}$, and $2\pi = \frac{2\pi F_s}{F_s}$. That gives us:

$$\cos \left( \frac{2\pi fn}{F_s} \right) = \cos \left( \frac{2\pi n(F_s - f)}{F_s} \right)$$

$$\cos \left( \frac{2\pi fn}{F_s} \right) = \cos \left( \frac{2\pi (f - F_s)n}{F_s} \right)$$

So a discrete-time cosine at frequency $f$ is also a cosine at frequency $F_s - f$, and it’s also a cosine at $f - F_s$. 
A discrete-time cosine at frequency $f$ is also a cosine at frequency $F_s - f$, and it's also a cosine at $f - F_s$.

So which of those frequencies will we hear when we play the sinusoid back again?

**ANSWER:** any frequency that can be reconstructed by the analog-to-digital converter. That means any frequency below the Nyquist frequency, $F_N = F_s/2$. 
Aliased Frequency

4Hz, at $F_s=9$ Hz, looks like 4Hz

4Hz, at $F_s=8$ Hz, looks like 4Hz

4Hz, at $F_s=7$ Hz, looks like 3Hz

4Hz, at $F_s=6$ Hz, looks like 2Hz

4Hz, at $F_s=5$ Hz, looks like 1Hz
Aliased Frequency

4Hz, at $F_s=4.5$ Hz, looks like 0.5Hz

4Hz, at $F_s=4$ Hz, looks like 0Hz

4Hz, at $F_s=3.5$ Hz, looks like 0.5Hz

4Hz, at $F_s=3$ Hz, looks like 1Hz

4Hz, at $F_s=2.5$ Hz, looks like 1Hz
Aliased Frequency

All of the following frequencies are actually the same frequency when a cosine is sampled at $F_s$ samples/second.

$$f_a \in \{f - \ell F_s, \ell F_s - f : \ell \in \text{any integer}\}$$

The “aliased frequency” is whichever of those is below Nyquist ($F_s/2$). Usually there’s only one that’s below Nyquist, so you can just look for

$$f_a = \min (f - \ell F_s, \ell F_s - f : \ell \in \text{any integer})$$
Outline

1. Review: Spectrum of continuous-time signals
2. Sampling
3. Aliasing
4. Aliased Frequency
5. Aliased Phase
6. Summary
7. Written Example
Sine waves are different for the following reason:

\[
\sin(\phi) = -\sin(2\pi n - \phi) \\
\sin(\phi) = \sin(\phi - 2\pi n)
\]
Therefore:

\[
\sin \left( \frac{2\pi fn}{F_s} \right) = -\sin \left( \frac{2\pi n(F_s - f)}{F_s} \right)
\]

\[
\sin \left( \frac{2\pi fn}{F_s} \right) = \sin \left( \frac{2\pi(f - F_s)n}{F_s} \right)
\]

So a discrete-time sine at frequency \( f \) is also a \textbf{negative} sine at frequency \( F_s - f \), and a \textbf{positive} sine at frequency \( f - F_s \).
Sine is Different

- 4Hz sine, at $F_s=10$ Hz, looks like 4Hz
- 4Hz sine, at $F_s=9$ Hz, looks like 4Hz
- 4Hz sine, at $F_s=8$ Hz, looks like 4Hz
- 4Hz sine, at $F_s=7$ Hz, looks like 3Hz
- 4Hz sine, at $F_s=6$ Hz, looks like 2Hz
Aliased Phase of a General Phasor

For a general complex exponential, we get:

\[ z e^{j\phi} = z e^{j(\phi - 2\pi n)} = \left( z^* e^{j(2\pi n - \phi)} \right)^* \]

Therefore:

\[ \Re \left\{ z e^{j \frac{2\pi fn}{F_s}} \right\} = \Re \left\{ z e^{j \frac{2\pi (f - F_s)n}{F_s}} \right\} = \Re \left\{ z^* e^{j \frac{2\pi (F_s - f)n}{F_s}} \right\} \]
Aliased Phase of a General Phasor

Suppose we have some frequency $f$, and we’re trying to find its aliased frequency $f_a$.

- Among the several possibilities, if $f_a = F_s - f$ is below Nyquist, then that’s the frequency we’ll hear. Its phasor will be the complex conjugate of the original phasor,

  $$z_a = z^*$$

- On the other hand, if $f_a = f - F_s$ is below Nyquist, then that’s the frequency we’ll hear. Its phasor will be the same as the phasor of the original sinusoid:

  $$z_a = z$$
Aliased Phase of a General Phasor

4Hz at $-\pi/4$ phase, at $F_s=30$ Hz, looks like 4Hz with phase of $-\pi/4$

4Hz at $-\pi/4$ phase, at $F_s=11$ Hz, looks like 4Hz with phase of $-\pi/4$

4Hz at $-\pi/4$ phase, at $F_s=10$ Hz, looks like 4Hz with phase of $-\pi/4$

4Hz at $-\pi/4$ phase, at $F_s=6$ Hz, looks like 2Hz with phase of $+\pi/4$

4Hz at $-\pi/4$ phase, at $F_s=5$ Hz, looks like 1Hz with phase of $+\pi/4$
Outline

1. Review: Spectrum of continuous-time signals
2. Sampling
3. Aliasing
4. Aliased Frequency
5. Aliased Phase
6. Summary
7. Written Example
A sampled sinusoid can be reconstructed perfectly if the
Nyquist criterion is met, \( f < \frac{F_s}{2} \).

If the Nyquist criterion is violated, then:

- If \( \frac{F_s}{2} < f < F_s \), then it will be aliased to
  \[
  f_a = F_s - f \\
  z_a = z^*
  \]
  i.e., the sign of all sines will be reversed.

- If \( F_s < f < \frac{3F_s}{2} \), then it will be aliased to
  \[
  f_a = f - F_s \\
  z_a = z
  \]
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2. Sampling
3. Aliasing
4. Aliased Frequency
5. Aliased Phase
6. Summary
7. Written Example
Sketch a sinusoid with some arbitrary phase (say, $-\pi/4$). Show where the samples are if it’s sampled:

- more than twice per period
- more than once per period, but less than twice per period
- less than once per period