Lecture 31: Second-Order IIR Filters

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ECE 401: Signal and Image Analysis

Outline

Review: Poles and Zeros

A first-order autoregressive filter,

$$y[n] = x[n] + bx[n-1] + ay[n-1],$$

has the impulse response and transfer function

$$h[n] = a^n u[n] + ba^{n-1}u[n-1] \leftrightarrow H(z) = \frac{1+bz^{-1}}{1-az^{-1}},$$

where a is called the **pole** of the filter, and $-b$ is called its **zero**.

Causality and Stability

- A filter is **causal** if and only if the output, $y[n]$, depends only an **current and past** values of the input, $x[n], x[n-1], x[n-2], \ldots$.
- A filter is **stable** if and only if **every** finite-valued input generates a finite-valued output. A causal first-order IIR filter is stable if and only if $|a| < 1$.

Review: Poles and Zeros

Suppose $H(z) = \frac{1+bz^{-1}}{1-az^{-1}}$, and $|a| < 1$. Now let's evaluate $|H(\omega)|$, by evaluating $|H(z)|$ at $z = e^{j\omega}$:

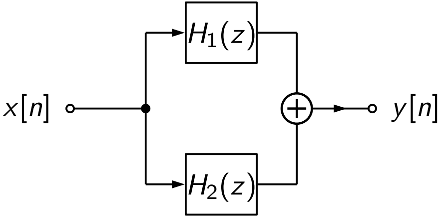
$$|H(\omega)| = \frac{|e^{j\omega} + b|}{|e^{j\omega} - a|}$$

What it means $|H(\omega)|$ is the ratio of two vector lengths:

- When the vector length $|e^{j\omega} + b|$ is small, then $|H(\omega)|$ is small.
- When $|e^{j\omega} - a|$ is small, then $|H(\omega)|$ is LARGE.

Review: Parallel Combination

Parallel combination of two systems looks like this:



Suppose that we know each of the systems separately:

$$H_1(z) = \frac{1}{1-p_1z^{-1}}, \qquad H_2(z) = \frac{1}{1-p_2z^{-1}}$$

Then, to get $H(z)$, we just have to add:

$$H(z) = \frac{1}{1-p_1z^{-1}} + \frac{1}{1-p_2z^{-1}} = \frac{2-(p_1+p_2)z^{-1}}{1-(p_1+p_2)z^{-1}+p_1p_2z^{-2}}$$

Outline

A General Second-Order All-Pole Filter

Let's construct a general second-order all-pole filter (leaving out the zeros; they're easy to add later).

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} = \frac{1}{1 - (p_1 + p_1^*)z^{-1} + p_1 p_1^* z^{-2}}$$

The difference equation that implements this filter is

$$Y(z) = X(z) + (p_1 + p_1^*)z^{-1}Y(z) - p_1 p_1^* z^{-2} Y(z)$$

Which converts to

$$y[n] = x[n] + 2\Re(p_1)y[n-1] - |p_1|^2 y[n-2]$$

Partial Fraction Expansion

In order to find the impulse response, we do a partial fraction expansion:

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} = \frac{C_1}{1 - p_1 z^{-1}} + \frac{C_1^*}{1 - p_1^* z^{-1}}$$

When we normalize the right-hand side of the equation above, we get the following in the numerator:

$$1 + 0 \times z^{-1} = C_1(1 - p_1^* z^{-1}) + C_1^*(1 - p_1 z^{-1})$$

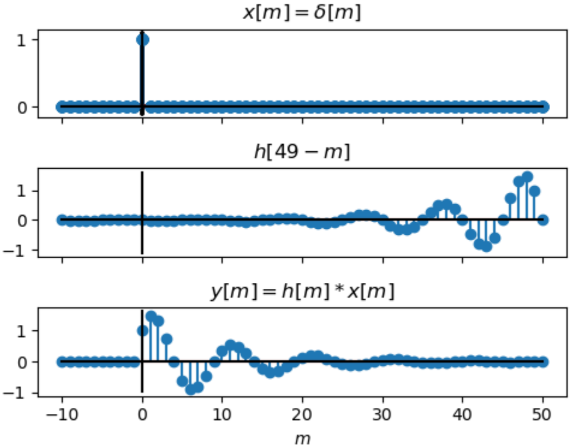
and therefore

$$C_1 = \frac{p_1}{p_1 - p_1^*}$$

Impulse Response of a Second-Order IIR

. . . and so we just inverse transform.

$$h[n] = C_1 p_1^n u[n] + C_1^*(p_1^*)^n u[n]$$



Understanding the Impulse Response of a Second-Order IIR

In order to **understand** the impulse response, maybe we should invent some more variables. Let's say that

$$p_1 = e^{-\sigma_1 + j\omega_1}, \quad p_1^* = e^{-\sigma_1 - j\omega_1}$$

where σ_1 is the half-bandwidth of the pole, and ω_1 is its center frequency. The partial fraction expansion gave us the constant

$$C_1 = \frac{p_1}{p_1 - p_1^*} = \frac{p_1}{e^{-\sigma_1}\left(e^{j\omega_1} - e^{-j\omega_1}\right)} = \frac{e^{j\omega_1}}{2j\sin(\omega_1)}$$

whose complex conjugate is

$$C_1^* = -\frac{e^{-j\omega_1}}{2j\sin(\omega_1)}$$

Impulse Response of a Second-Order IIR

Plugging in to the impulse response, we get

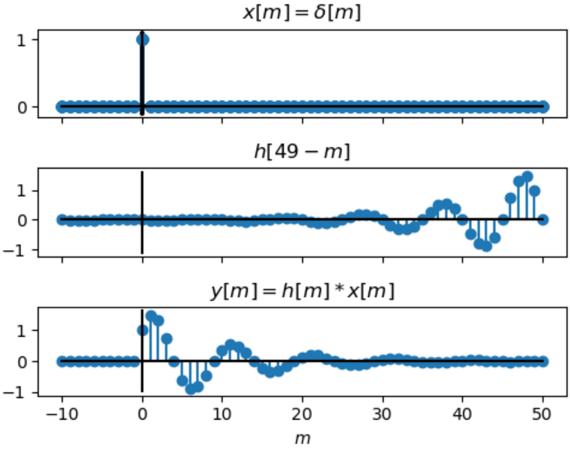
$$h[n] = \frac{1}{2j\sin(\omega_1)}\left(e^{j\omega_1}e^{(-\sigma_1+j\omega_1)n} - e^{-j\omega_1}e^{(-\sigma_1-j\omega_1)n}\right)u[n]$$

$$= \frac{1}{2j\sin(\omega_1)}e^{-\sigma_1 n}\left(e^{j\omega_1(n+1)} - e^{-j\omega_1(n+1)}\right)u[n]$$

$$= \frac{1}{\sin(\omega_1)}e^{-\sigma_1 n}\sin(\omega_1(n+1))u[n]$$

Impulse Response of a Second-Order IIR

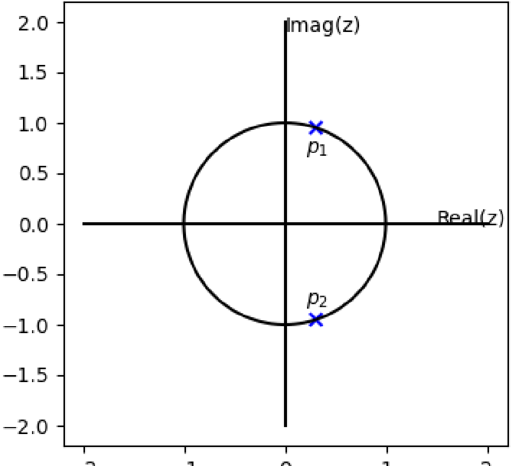
$$h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1(n+1)) u[n]$$



Outline

Example: Ideal Resonator

As the first example, let's suppose we put p_1 right on the unit circle, $p_1 = e^{j\omega_1}$.



Example: Resonator

The system function for this filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2\cos(\omega_1)z^{-1} + z^{-2}}$$

Solving for $y[n]$, we get the difference equation:

$$y[n] = x[n] + 2\cos(\omega_1)y[n-1] - y[n-2]$$

Example: Ideal Resonator

Just to make it concrete, let's choose $\omega_1 = \frac{\pi}{4}$, so the difference equation is

$$y[n] = x[n] + \sqrt{2}y[n-1] - y[n-2]$$

If we plug $x[n] = \delta[n]$ into this equation, we get

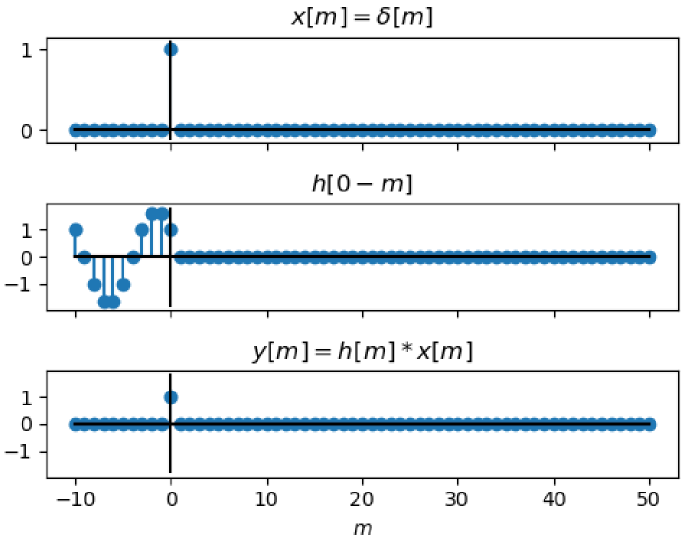
$$\begin{aligned} y[0] &= 1 \\ y[1] &= \sqrt{2} \\ y[2] &= 2 - 1 = 1 \\ y[3] &= \sqrt{2} - \sqrt{2} = 0 \\ y[4] &= -1 \\ y[5] &= -\sqrt{2} \\ &\vdots \end{aligned}$$

Example: Ideal Resonator

Putting $p_1 = e^{j\omega_1}$ into the general form, we find that the impulse response of this filter is

$$h[n] = \frac{1}{\sin(\omega_1)} \sin(\omega_1(n+1))u[n]$$

This is called an "ideal resonator" because it keeps ringing forever.



$x[m]=\delta[m]$
1
0
$h[0-m]$
1
0
−1
$y[m]=h[m]*x[m]$
1
0
−1
−10
0
10
20
30
40
50
m

An Ideal Resonator is Unstable

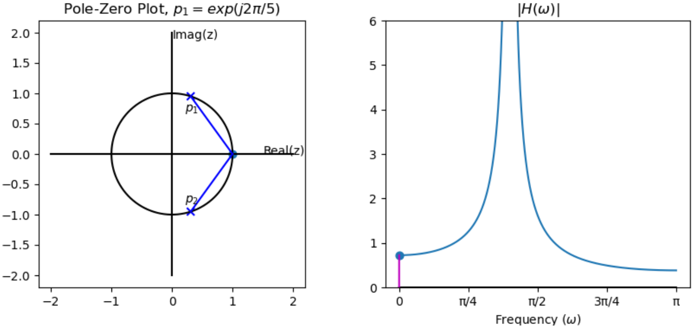
A resonator is unstable. The easiest way to see what this means is by looking at its frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}} = \frac{1}{(1-e^{j(\omega_1-\omega)})(1-e^{j(-\omega_1-\omega)})}$$

$$H(\omega_1) = \frac{1}{(1-1)(1-e^{-2j\omega_1})} = \infty$$

So if $x[n] = \cos(\omega_1 n)$, then $y[n]$ is

$$y[n] = |H(\omega_1)| \cos\left(\omega_1 n + \angle H(\omega_1)\right) = \infty$$


Pole-Zero Plot, $p_1 = exp(j2\pi/5)$
Imag(z)
Real(z)
p_1
p_2
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
$|H(\omega)|$
6
5
4
3
2
1
0
0
π/4
π/2
3π/4
π
Frequency (ω)

Instability from the POV of the Impulse Response

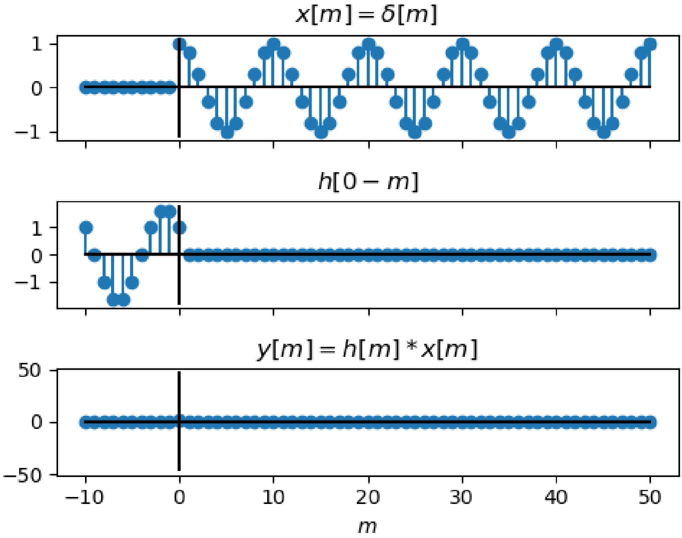
From the point of view of the impulse response, you can think of instability like this:

$$y[n] = \sum_m x[m]h[n-m]$$

Suppose $x[m] = \cos(\omega_1 m)u[m]$. Then

$$y[n] = x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \dots$$

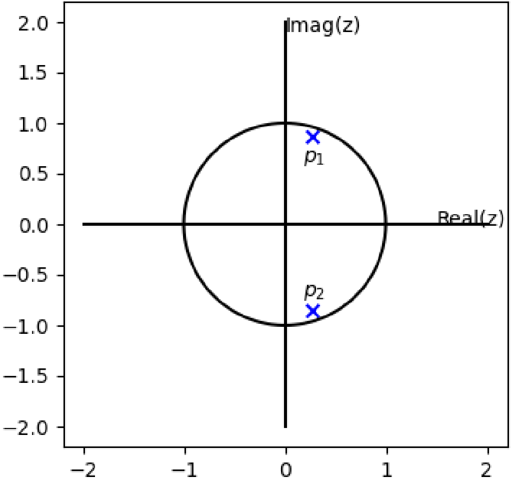
We keep adding extra copies of $h[n-m]$, for each m, forever. Since $h[n]$ never dies away, the result is that we keep building up $y[n]$ toward infinity.


$x[m] = \delta[m]$
1
0
−1
$h[0 - m]$
1
0
−1
$y[m] = h[m] * x[m]$
50
0
−50
−10
0
10
20
30
40
50
m

Outline

Example: Stable Resonator

Now, let's suppose we put p_1 inside the unit circle, $p_1 = e^{-\sigma_1 + j\omega_1}$.



Example: Stable Resonator

The system function for this filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2e^{-\sigma_1}\cos(\omega_1)z^{-1} + e^{-2\sigma_1}z^{-2}}$$

Solving for $y[n]$, we get the difference equation:

$$y[n] = x[n] + 2e^{-\sigma_1}\cos(\omega_1)y[n-1] - e^{-2\sigma_1}y[n-2]$$

Example: Stable Resonator

Just to make it concrete, let's choose $\omega_1 = \frac{\pi}{4}$, and $e^{-\sigma_1} = 0.9$, so the difference equation is

$$y[n] = x[n] + 0.9\sqrt{2}y[n-1] - 0.81y[n-2]$$

If we plug $x[n] = \delta[n]$ into this equation, we get

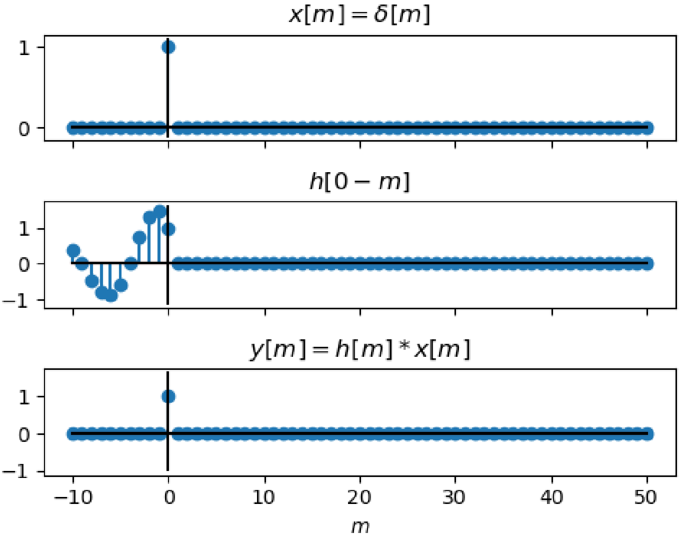
$$\begin{aligned}
y[0] &= 1 \\
y[1] &= 0.9\sqrt{2} \\
y[2] &= (0.9\sqrt{2})^2 - 0.81 = 0.81 \\
y[3] &= (0.9\sqrt{2})(0.81) - (0.81)(0.9\sqrt{2}) = 0 \\
y[4] &= -(0.81)^2 \\
y[5] &= -(0.9\sqrt{2})(0.81)^2 \\
&\vdots
\end{aligned}$$

Example: Stable Resonator

Putting $p_1 = e^{-\sigma_1 + j\omega_1}$ into the general form, we find that the impulse response of this filter is

$$h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1 (n+1)) u[n]$$

This is called a "stable resonator" or a "stable sinusoid" or a "damped resonator" or a "damped sinusoid." It rings at the frequency ω_1, but it gradually decays away.


$x[m] = \delta[m]$
1
0
$h[0-m]$
1
0
−1
$y[m] = h[m] * x[m]$
1
0
−1
−10
0
10
20
30
40
50
m

A Damped Resonator is Stable

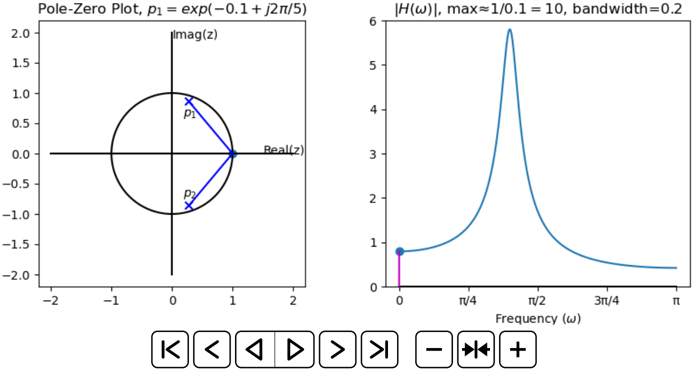
A damped resonator is stable: any finite input will generate a finite output.

$$H(\omega) = H(z)\big|_{z=e^{j\omega}} = \frac{1}{(1-e^{-\sigma_1+j(\omega_1-\omega)})(1-e^{-\sigma_1+j(-\omega_1-\omega)})}$$

$$H(\omega_1) = \frac{1}{(1-e^{-\sigma_1})(1-e^{-\sigma_1-2j\omega_1})} \approx \frac{1}{1-e^{-\sigma_1}} \approx \frac{1}{\sigma_1}$$

So if $x[n] = \cos(\omega_1 n)$, then $y[n]$ is

$$\begin{aligned} y[n] &= |H(\omega_1)| \cos\left(\omega_1 n + \angle H(\omega_1)\right) \\ &\approx \frac{1}{\sigma_1} \cos\left(\omega_1 n + \angle H(\omega_1)\right) \end{aligned}$$


Pole-Zero Plot, $p_1 = exp(-0.1 + j2\pi/5)$
Imag(z)
Real(z)
p_1
p_2
$|H(\omega)|$, max≈1/0.1 = 10, bandwidth=0.2
Frequency (ω)
0
π/4
π/2
3π/4
π

Stability from the POV of the Impulse Response

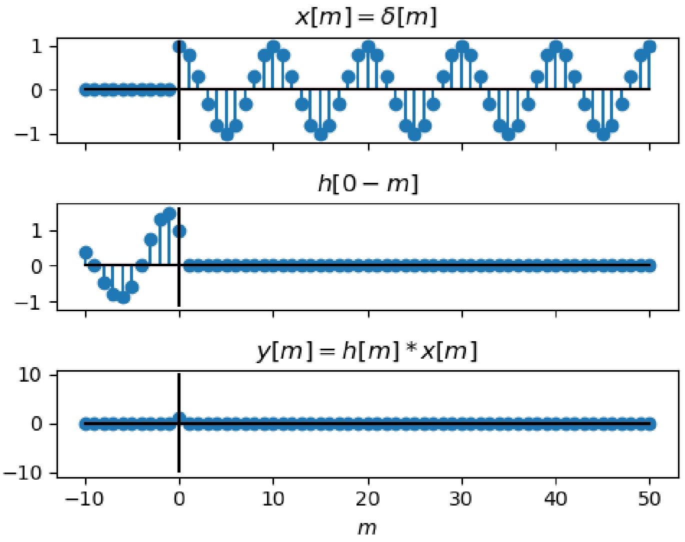
From the point of view of the impulse response, you can think of stability like this:

$$y[n] = \sum_m x[m]h[n-m]$$

Suppose $x[m] = \cos(\omega_1 m)u[m]$. Then

$$y[n] = x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \ldots$$

We keep adding extra copies of $h[n-m]$, for each m, forever. However, since each $h[n-m]$ dies away, and since they are being added with a time delay between them, the result never builds all the way to infinity.



x[m] = δ[m]
h[0 − m]
y[m] = h[m] * x[m]
1
0
−1
10
−10
−10 0 10 20 30 40 50
m

Outline

Magnitude Response of an All-Pole Filter

Until now, I have often used this trick, but have never really discussed it with you:

$$
\begin{aligned}
|H(z)| &= \frac{1}{|1-p_1z^{-1}| \times |1-p_2z^{-1}|}\\
&= \frac{|z|^2}{|z-p_1| \times |z-p_2|}\\
&= \frac{1}{|e^{j\omega}-p_1| \times |e^{j\omega}-p_2|}
\end{aligned}
$$

That's why the magnitude response is just one over the product of the two vector lengths.

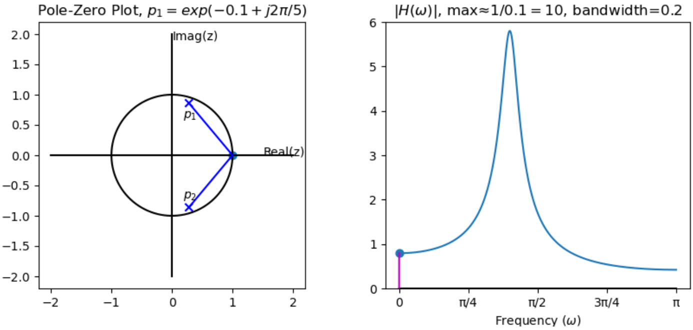
Magnitude Response at $\omega = \omega_1 \pm \epsilon$

Now let's suppose $p_1 = e^{-\sigma_1 + j\omega_1}$, and $p_2 = p_1^* = e^{-\sigma_1 - j\omega_1}$. Consider what happens when $\omega = \omega_1 \pm \epsilon$ for small values of ϵ.

- There are two poles, one at ω_1, one at $-\omega_1$.
- The pole at $-\omega_1$ is very far away from $\omega \approx +\omega_1$. In fact, over the whole range $\omega = \omega_1 \pm \epsilon$, this distance remains approximately constant:

$$\begin{aligned} |e^{j\omega} - p_1^*| &= |e^{j(\omega_1 \pm \epsilon)} - e^{-\sigma_1 - j\omega_1)}| \\ &\approx |e^{j\omega_1} - e^{-j\omega_1}| \\ &= 2|\sin(\omega_1)| \end{aligned}$$

One pole remains very far away:



Magnitude Response at $\omega = \omega_1 \pm \epsilon$

The other vector is the one that decides the shape of $|H(\omega)|$. We could write it in a few different ways:

$$\begin{aligned}
|e^{j\omega} - p_1| &= |e^{j\omega}| \times |1 - p_1 e^{-j\omega}| \\
&= 1 \times |1 - p_1 e^{-j\omega}| \\
&= 1 \times |1 - e^{-\sigma_1 + j\omega_1} e^{-j\omega}| \\
&= 1 \times |1 - e^{-\sigma_1 + j\omega_1} e^{-j(\omega_1 \pm \epsilon)}| \\
&= 1 \times |1 - e^{-\sigma_1 \pm j\epsilon}|
\end{aligned}$$

Let's use the approximation $e^x \approx 1 + x$, which is true for small values of x. That gives us

$$|e^{j\omega} - p_1| = | - \sigma_1 \pm j\epsilon|$$

Magnitude Response at $\omega = \omega_1 \pm \epsilon$

There are three frequencies that really matter:

1. Right at the pole, at $\omega = \omega_1$, we have

$$|e^{j\omega} - p_1| = \sigma_1$$

2. At $\pm$ half a bandwidth, $\omega = \omega_1 \pm \sigma_1$, we have

$$|e^{j\omega} - p_1| = |-\sigma_1 \mp j\sigma_1| = \sigma_1\sqrt{2}$$

Magnitude Response at $\omega = \omega_1 \pm \epsilon$

There are three frequencies that really matter:

1. Right at the pole, at $\omega = \omega_1$, we have

$$|H(\omega_1)| \propto \frac{1}{\sigma_1}$$

2. At $\pm$ half a bandwidth, $\omega = \omega_1 \pm \sigma_1$, we have

$$|H(\omega_1 \pm \sigma_1)| = \frac{1}{\sqrt{2}} |H(\omega_1)|$$

3dB Bandwidth

- The 3dB bandwidth of an all-pole filter is the width of the peak, measured at a level $1/\sqrt{2}$ relative to its peak.
- σ_1 is half the bandwidth.

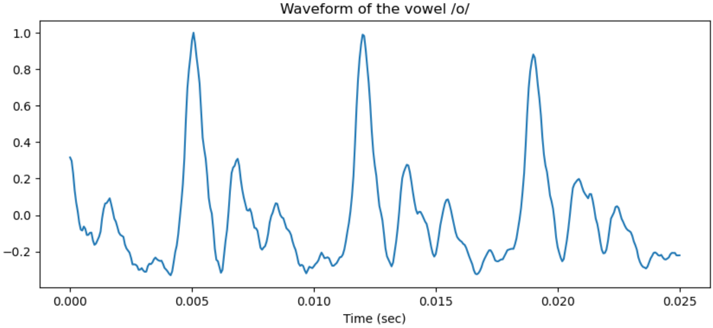
Outline

Speech

The most important example of a damped resonator is speech.

- Once every 5-10ms, your vocal folds close, abruptly shutting off the airflow. This causes an instantaneous pressure impulse.
- The impulse activates the impulse response of your vocal tract (the area between the glottis and the lips).
- Your vocal tract is a damped resonator.

Speech is made up of Damped Sinusoids



Speech is made up of Damped Sinusoids

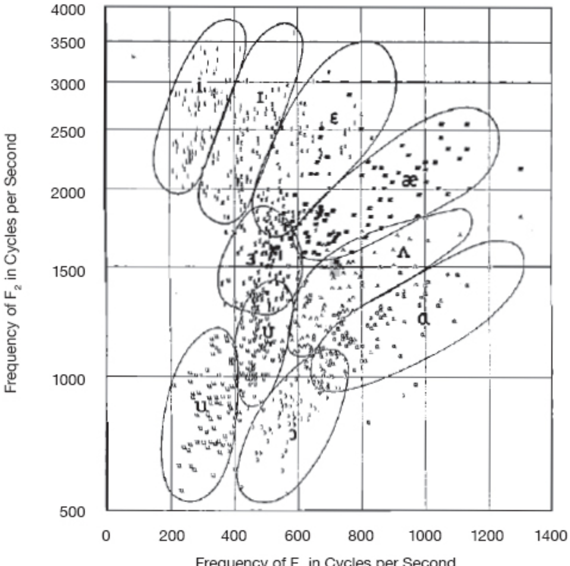
Your vocal tract has an infinite number of resonant frequencies, all of which ring at once:

$$H(z) = \prod_{k=1}^{\infty} \frac{1}{(1 - p_k z^{-1})(1 - p_k^* z^{-1})}$$

There are an infinite number, but most are VERY heavily damped, so usually we only hear the first three or four.

Center Freqs of First Two Poles Specify the Vowel

(Peterson & Barney, 1952)



First Formant Resonator

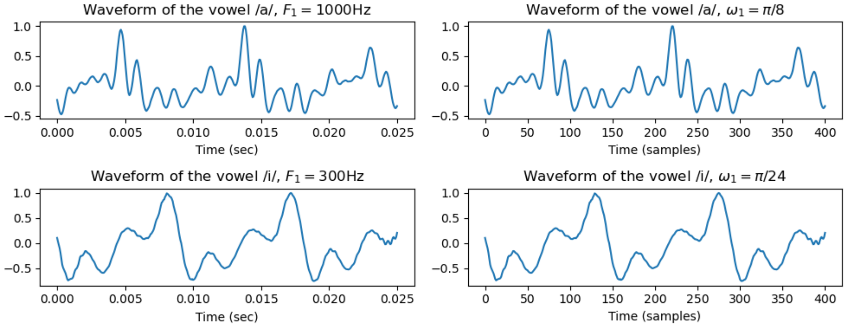
When you look at a speech waveform, $x[n]$, most of what you see is the first resonance, called the "first formant." Its resonant frequency is roughly $400 \leq F_1 \leq 800$ usually, so at $F_s = 16000$Hz sampling frequency, we get

$$\omega_1 = \frac{2\pi F_1}{F_S} \in \left[\frac{\pi}{20}, \frac{\pi}{10}\right]$$

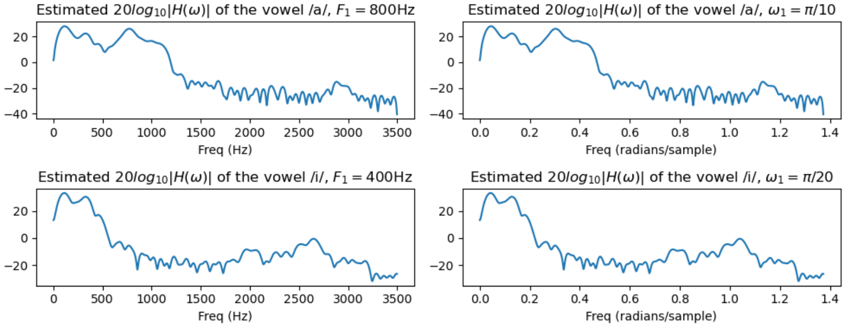
Its bandwidth might be about $B_1 \approx 400$Hz, so

$$\sigma_1 = \frac{1}{2}\left(\frac{2\pi B_1}{F_s}\right) \approx \frac{\pi}{40}$$

First Formant Frequency and Bandwidth in the Waveform



First Formant Frequency and Bandwidth in the Spectrum



Outline

Impulse Response of a Second-Order All-Pole Filter

A general all-pole filter has the system function

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} = \frac{1}{1 - (p_1 + p_1^*)z^{-1} + p_1 p_1^* z^{-2}}$$

Its impulse response is

$$h[n] = C_1 p_1^n u[n] + C_1^* (p_1^*)^n u[n]$$

Impulse Response of a Second-Order All-Pole Filter

We can take advantage of complex numbers to write these as

$$H(z) = \frac{1}{1 - 2e^{-\sigma_1}\cos(\omega_1)z^{-1} + e^{-2\sigma_1}z^{-2}}$$

and

$$h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1(n+1))u[n]$$

Magnitude Response of a Second-Order All-Pole Filter

In the frequency response, there are three frequencies that really matter:

1. Right at the pole, at $\omega = \omega_1$, we have

$$|H(\omega_1)| \propto \frac{1}{\sigma_1}$$

2. At $\pm$ half a bandwidth, $\omega = \omega_1 \pm \sigma_1$, we have

$$|H(\omega_1 \pm \sigma_1)| = \frac{1}{\sqrt{2}}|H(\omega_1)|$$