Lecture 30: Block Diagrams and the Inverse Z Transform

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ECE 401: Signal and Image Analysis
1. Review: FIR and IIR Filters, and System Functions

2. The System Function and Block Diagrams

3. Inverse Z Transform

4. Summary

5. Written Example
Outline

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An autoregressive filter is also called **infinite impulse response (IIR)**, because $h[n]$ has infinite length.

A filter with only feedforward coefficients, and no feedback coefficients, is called **finite impulse response (FIR)**, because $h[n]$ has finite length (its length is just the number of feedforward terms in the difference equation).
A first-order autoregressive filter,

$$y[n] = x[n] + bx[n - 1] + ay[n - 1],$$

has the impulse response and system function

$$h[n] = a^n u[n] + ba^{n-1} u[n - 1] \leftrightarrow H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}},$$

where $a$ is called the **pole** of the filter, and $-b$ is called its **zero**.
A filter is **causal** if and only if the output, $y[n]$, depends only on **current and past** values of the input, $x[n], x[n-1], x[n-2], \ldots$.

A filter is **stable** if and only if **every** finite-valued input generates a finite-valued output. A causal first-order IIR filter is stable if and only if $|a| < 1$. 
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Why use block diagrams?

A first-order difference equation looks like

\[ y[n] = b_0 x[n] + b_1 x[n - 1] + ay[n - 1] \]

- It's pretty easy to understand what computation is taking place in a first-order difference equation.
- As we get to higher-order systems, though, the equations for implementing them will be kind of complicated.
- In order to make the complicated equations very easy, we represent the equations using block diagrams.
Elements of a block diagram

A block diagram has just three main element types:

1. **Multiplier:** the following element means $y[n] = b_0 x[n]$:
   
   ![Multiplier Diagram](image)
   
   $y[n] = b_0 x[n]$

2. **Unit Delay:** the following element means $y[n] = x[n - 1]$ (i.e., $Y(z) = z^{-1}X(z)$):
   
   ![Unit Delay Diagram](image)
   
   $y[n] = z^{-1} x[n]$

3. **Adder:** the following element means $z[n] = x[n] + y[n]$:
   
   ![Adder Diagram](image)
   
   $z[n] = x[n] + y[n]$
Example: Time Domain

Here's an example of a complete block diagram:

\[ x[n] \quad + \quad y[n] \]

This block diagram is equivalent to the following equation:

\[ y[n] = x[n] + ay[n - 1] \]

Notice that we can read it, also, as

\[ Y(z) = X(z) + az^{-1}Y(z) \quad \Rightarrow \quad H(z) = \frac{1}{1 - az^{-1}} \]
Now consider how we can represent a complete first-order IIR filter, including both the pole and the zero. Here it is in the $z$-domain:

$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + a_1 z^{-1} Y(z).$$

When we implement it, we would write a line of Python that does this:

$$y[n] = b_0 x[n] + b_1 x[n - 1] + a_1 y[n - 1],$$

which is exactly this block diagram:
Now let’s talk about how to combine systems.

- **Series combination**: passing the signal through two systems in series is like multiplying the system functions:

\[ H(z) = H_2(z)H_1(z) \]

- **Parallel combination**: passing the signal through two systems in parallel, then adding the outputs, is like adding the system functions:

\[ H(z) = H_1(z) + H_2(z) \]
One Block for Each System

Suppose that one of the two systems, $H_1(z)$, looks like this:

\[ x[n] \rightarrow + \rightarrow y[n] \]

\[ \begin{align*}
    \frac{1}{1 - p_1 z^{-1}} \\
    z^{-1}
\end{align*} \]

and has the system function

\[ H_1(z) = \frac{1}{1 - p_1 z^{-1}} \]

Let’s represent the whole system using a single box:

\[ x[n] \rightarrow H_1(z) \rightarrow y[n] \]
The series combination, then, looks like this:

\[ x[n] \xrightarrow{H_1(z)} v[n] \xrightarrow{H_2(z)} y[n] \]

This means that

\[ Y(z) = H_2(z)V(z) = H_2(z)H_1(z)X(z) \]

and therefore

\[ H(z) = \frac{Y(z)}{X(z)} = H_1(z)H_2(z) \]
Series Combination

The series combination, then, looks like this:

\[ x[n] \xrightarrow{} H_1(z) \xrightarrow{} H_2(z) \xrightarrow{} y_2[n] \]

Suppose that we know each of the systems separately:

\[ H_1(z) = \frac{1}{1 - p_1 z^{-1}}, \quad H_2(z) = \frac{1}{1 - p_2 z^{-1}} \]

Then, to get \( H(z) \), we just have to multiply:

\[ H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} = \frac{1}{1 - (p_1 + p_2) z^{-1} + p_1 p_2 z^{-2}} \]
Parallel combination of two systems looks like this:

![Block Diagram](image)

This means that

\[ Y(z) = H_1(z)X(z) + H_2(z)X(z) \]

and therefore

\[ H(z) = \frac{Y(z)}{X(z)} = H_1(z) + H_2(z) \]
Parallel combination of two systems looks like this:

\[
\begin{align*}
H_1(z) &= \frac{1}{1 - p_1 z^{-1}} \\
H_2(z) &= \frac{1}{1 - p_2 z^{-1}} \\
\end{align*}
\]

Suppose that we know each of the systems separately:

\[
H_1(z) = \frac{1}{1 - p_1 z^{-1}}, \quad H_2(z) = \frac{1}{1 - p_2 z^{-1}}
\]

Then, to get \( H(z) \), we just have to add:

\[
H(z) = \frac{1}{1 - p_1 z^{-1}} + \frac{1}{1 - p_2 z^{-1}} = \frac{2 - (p_1 + p_2) z^{-1}}{1 - (p_1 + p_2) z^{-1} + p_1 p_2 z^{-2}}
\]
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Suppose you know $H(z)$, and you want to find $h[n]$. How can you do that?
How to find the inverse Z transform

Any IIR filter $H(z)$ can be written as...

- **denominator terms**, each with this form:

  $$G_\ell(z) = \frac{1}{1 - az^{-1}} \quad \leftrightarrow \quad g_\ell[n] = a^n u[n],$$

- each possibly multiplied by a **numerator** term, like this one:

  $$D_k(z) = b_k z^{-k} \quad \leftrightarrow \quad d_k[n] = b_k \delta[n - k].$$
Step #1: Numerator Terms

Consider one that you already know:

\[ H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}} = \left( \frac{1}{1 - az^{-1}} \right) + bz^{-1} \left( \frac{1}{1 - az^{-1}} \right) \]

and therefore

\[ h[n] = (a^n u[n]) + b (a^{n-1} u[n-1]) \]
So here is the inverse transform of $H(z) = \frac{1+0.5z^{-1}}{1-0.85z^{-1}}$:
Step #1: Numerator Terms

In general, if

\[ G(z) = \frac{1}{A(z)} \]

for any polynomial \( A(z) \), and

\[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{A(z)} \]

then

\[ h[n] = b_0 g[n] + b_1 g[n - 1] + \cdots + b_M g[n - M] \]
Now we need to figure out the inverse transform of

\[ G(z) = \frac{1}{A(z)} \]

We will solve this using a method called **partial fraction expansion.**
Step #2: Partial Fraction Expansion

Partial fraction expansion works like this:

1. Factor $A(z)$:

   \[ G(z) = \frac{1}{\prod_{\ell=1}^{N} (1 - p_{\ell}z^{-1})} \]

2. Assume that $G(z)$ is the result of a parallel system combination:

   \[ G(z) = \frac{C_1}{1 - p_1z^{-1}} + \frac{C_2}{1 - p_2z^{-1}} + \cdots \]

3. Find the constants, $C_{\ell}$, that make the equation true. Such constants always exist, as long as none of the roots are repeated ($p_k \neq p_{\ell}$ for $k \neq \ell$).
Partial Fraction Expansion: Example

Step # 1: Factor it:

\[
\frac{1}{1 - 1.2z^{-1} + 0.72z^{-2}} = \frac{1}{(1 - (0.6 + j0.6)z^{-1})(1 - (0.6 - j0.6)z^{-1})}
\]

Step #2: Express it as a sum:

\[
\frac{1}{1 - 1.2z^{-1} + 0.72z^{-2}} = \frac{C_1}{1 - (0.6 + j0.6)z^{-1}} + \frac{C_2}{1 - (0.6 - j0.6)z^{-1}}
\]

Step #3: Find the constants. The algebra is annoying, but it turns out that:

\[
C_1 = \frac{1}{2} - j\frac{1}{2}, \quad C_2 = \frac{1}{2} + j\frac{1}{2}
\]
Partial Fraction Expansion: Example

The system function is:

\[
G(z) = \frac{1}{1 - 1.2z^{-1} + 0.72z^{-2}} = \frac{0.5 - 0.5j}{1 - (0.6 + j0.6)z^{-1}} + \frac{0.5 + 0.5j}{1 - (0.6 - j0.6)z^{-1}}
\]

and therefore the impulse response is:

\[
g[n] = (0.5 - 0.5j)(0.6 + 0.6j)^n u[n] + (0.5 + 0.5j)(0.6 - j0.6)^n u[n]
= \left(0.5\sqrt{2}e^{-j\frac{\pi}{4}} \left(0.6\sqrt{2}e^{j\frac{\pi}{4}}\right)^n + 0.5\sqrt{2}e^{j\frac{\pi}{4}} \left(0.6\sqrt{2}e^{-j\frac{\pi}{4}}\right)^n\right) u[n]
= \sqrt{2}(0.6\sqrt{2})^n \cos \left(\frac{\pi}{4}(n - 1)\right) u[n]
\]
\[ g_1[n] = (0.5 - 0.5j)(0.6 + 0.6j)^n u[n] \text{ (imaginary part dashed)} \]

\[ g_2[n] = (0.5 + 0.5j)(0.6 - 0.6j)^n u[n] \text{ (imaginary part dashed)} \]

\[ g_1[n] + g_2[n] = (0.5\sqrt{2})(0.6\sqrt{2})^n \cos(\pi(n - 1)/4) u[n] \]
How to find the inverse Z transform

Any IIR filter $H(z)$ can be written as...

- a partial fraction expansion into a sum of denominator terms, each with this form:

$$G_\ell(z) = \frac{1}{1 - az^{-1}} \iff g_\ell[n] = a^n u[n],$$

- each possibly multiplied by a numerator term, like this one:

$$D_k(z) = b_k z^{-k} \iff d_k[n] = b_k \delta[n - k].$$
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Summary: Block Diagrams

- A **block diagram** shows the delays, additions, and multiplications necessary to compute output from input.

- **Series combination**: passing the signal through two systems **in series** is like multiplying the system functions:

\[ H(z) = H_2(z)H_1(z) \]

- **Parallel combination**: passing the signal through two systems **in parallel**, then adding the outputs, is like adding the system functions:

\[ H(z) = H_1(z) + H_2(z) \]
Summary: Inverse Z Transform

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Find the inverse Z transform of

\[ H(z) = \frac{1 - 0.7z^{-1}}{1 - 0.81z^{-2}} \]