Lecture 26: DTFT of a Sinusoid

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ECE 401: Signal and Image Analysis, Fall 2022

Outline

Review: DFT, DTFT, and Fourier Series

Magnitude-summable signals have a DTFT:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Periodic signals have a Fourier series:

$$X_k = \frac{1}{N}\sum_{n=1}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}, \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}}$$

Finite-length or periodic signals have a DFT:

$$X[k] = \sum_{n=1}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}, \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}$$

Review: DFT of a Sinusoid

To find the DFT of a sinusoid, we use the frequency-shift property of the DFT:

$$x[n] = \cos(\omega_0 n) w[n] = \left(\frac{1}{2} w[n] e^{j\omega_0 n} + \frac{1}{2} w[n] e^{-j\omega_0 n} \right)$$

$$\leftrightarrow$$

$$X[k] = \frac{1}{2} W\left(\frac{2\pi k}{N} - \omega_0 \right) + \frac{1}{2} W\left(\frac{2\pi k}{N} + \omega_0 \right)$$

where $W(\omega)$ is the DTFT of the window.

Today's Questions

Today's questions are:

1. Can we use the frequency-shift property to find the DTFT of a windowed sinusoid?
2. Can we use something like that to find the DTFT of a non-windowed, infinite length sinusoid?

Outline

DTFT of a Windowed Sinusoid

First, let's find the DTFT of a windowed sinusoid. This is easy; it's the same as the DFT. Since

$$x[n] = \cos(\omega_0 n)w[n] = \left(\frac{1}{2}w[n]e^{j\omega_0 n} + \frac{1}{2}w[n]e^{-j\omega_0 n}\right)$$

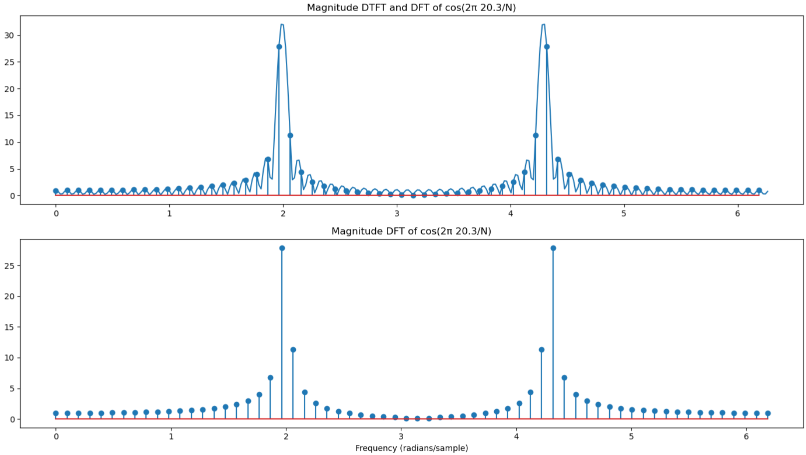
We can just use the frequency-shift property of the DTFT to get

$$X(\omega) = \frac{1}{2}W(\omega - \omega_0) + \frac{1}{2}W(\omega + \omega_0)$$

DFT of a Cosine

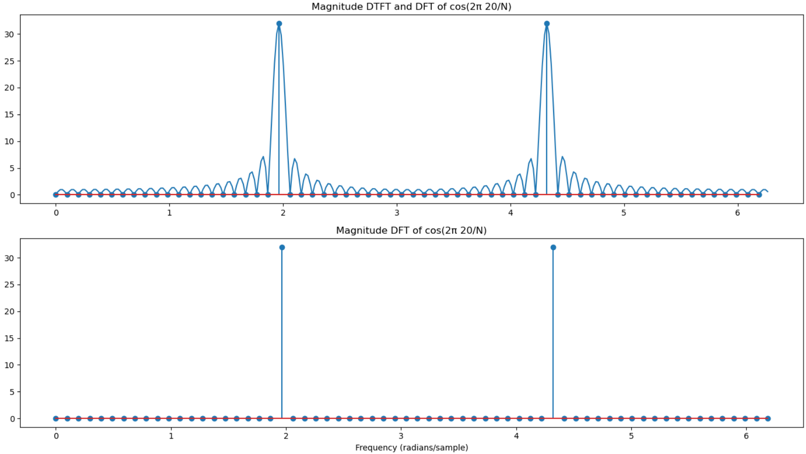
Here are the DTFT and DFT of

$$x[n] = \cos\left(\frac{2\pi 20.3}{N}n\right) w[n]$$



DFT of a Cosine

Here are the DTFT and DFT of a cosine at a frequency that's a multiple of $2\pi k/N$.



Outline

DTFT of a Non-Windowed Sinusoid

- How about $x[n] = \cos(\omega_0 n)$, with no windows? Does it have a DTFT?
- It's not magnitude-summable!

$$\sum_{n=-\infty}^{\infty} |x[n]| = \infty$$

Therefore, there's no guarantee that it has a valid DTFT.

- In fact, we will need to make up some new math in order to find the DTFT of this signal.

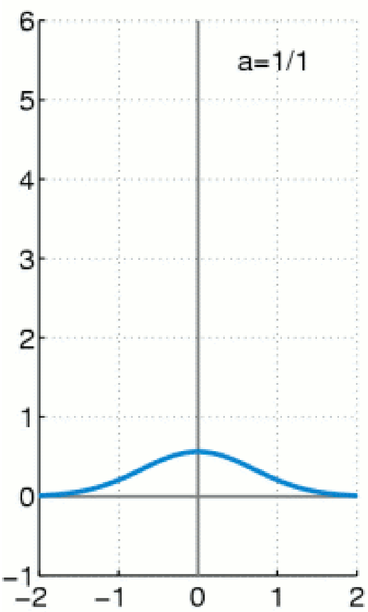
The Dirac Delta Function

The Dirac delta function, $\delta(\omega)$, is defined as:

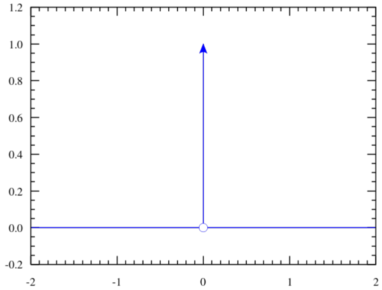
- $\delta(\omega) = 0$ for all ω other than $\omega = 0$.
- $\delta(0) = \infty$
- The integral of $\delta(\omega)$, from any negative ω to any positive ω, is exactly 1:

$$\int_{-\epsilon}^{\epsilon} \delta(\omega) d\omega = 1$$

It's useful to imagine the Dirac delta function as a tall, thin function — a Gaussian, a rectangle, or whatever — with zero width, infinite height, and an area of exactly 1.



We usually draw it like this. The arrow has zero width, infinite height, and an area of exactly 1.0.



Integrating a Dirac Delta

The key use of a Dirac delta is that, when we multiply it by any function and integrate,

- All the values of that function at $\omega \neq 0$ are multiplied by $\delta(\omega) = 0$
- The value at $\omega = 0$ is multiplied by $+\infty$, in such a way that the integral is exactly:

$$\int_{-\pi}^{\pi} f(\omega)\delta(\omega)d\omega = f(0)$$

Integrating a Shifted Dirac Delta

The delta function can also be shifted, to some frequency ω_0. This is written as $\delta(\omega - \omega_0)$.

- All the values of that function at $\omega \neq \omega_0$ are multiplied by $\delta(\omega - \omega_0) = 0$
- The value at $\omega = \omega_0$ is multiplied by $+\infty$, in such a way that the integral is exactly:

$$\int_{-\pi}^{\pi} f(\omega)\delta(\omega - \omega_0) d\omega = f(\omega_0)$$

Inverse DTFT of a Shifted Dirac Delta

Thus, for example,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(\omega-\omega_0)e^{j\omega n}d\omega = \frac{1}{2\pi}e^{j\omega_0 n}$$

In other words, the inverse DTFT of $Y(\omega)=\delta(\omega-\omega_0)$ is $y[n]=\frac{1}{2\pi}e^{j\omega_0 n}$, a complex exponential.

DTFT Pairs

By the linearity of the DTFT, we therefore have the following useful DTFT pairs:

$$e^{j\omega_0 n} \quad \leftrightarrow \quad 2\pi\delta(\omega - \omega_0),$$

and

$$\cos(\omega_0 n) \quad \leftrightarrow \quad \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Why This Answer Makes Sense

Suppose we were to try to find the DTFT of $x[n] = e^{j\omega_0 n}$ directly:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j(\omega-\omega_0)n}$$

- At frequencies $\omega \neq \omega_0$, we would be adding the samples of a sinusoid, which would give us $X(\omega) = 0$.
- At $\omega = \omega_0$, the summation becomes

$$X(\omega_0) = \sum_{n=-\infty}^{\infty} 1 = \infty$$

- So $X(\omega_0) = \infty$, and $X(\omega) = 0$ everywhere else. So it's a Dirac delta! The only thing the forward transform **doesn't** tell us is: **what kind of infinity?**

Why This Answer Makes Sense

- So $X(\omega_0)=\infty$, and $X(\omega)=0$ everywhere else. So it's a Dirac delta! The only thing the forward transform **doesn't** tell us is: **what kind of infinity?**
- The inverse DTFT gives us the answer. It needs to be the kind of infinity such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\omega)e^{j\omega n}d\omega=e^{j\omega_0 n},$$

and the solution is $X(\omega)=2\pi\delta(\omega-\omega_0)$

Outline

Windowing in Time = Convolution in Frequency

Remember that windowing in time = convolution in frequency:

$$y[n] = x[n]w[n] \quad \leftrightarrow \quad Y(\omega) = \frac{1}{2\pi}X(\omega) * W(\omega).$$

But if $x[n] = \cos(\omega_0 n)$, we already know that

$$y[n] = \cos(\omega_0 n)w[n] \quad \leftrightarrow \quad Y(\omega) = \frac{1}{2}W(\omega - \omega_0) + \frac{1}{2}W(\omega + \omega_0)$$

Can we reconcile these two facts?

Convolving with a Dirac delta function

The delta function is defined by this sampling property:

$$\int_{-\pi}^{\pi} \delta(\omega - \omega_0) f(\omega) d\omega = f(\omega_0)$$

What does that mean about convolution? Let's try it:

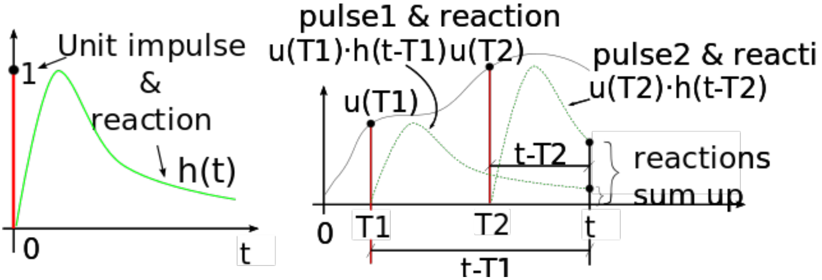
$$\delta(\omega - \omega_0) * W(\omega) = \int_{-\pi}^{\pi} \delta(\theta - \omega_0) W(\omega - \theta) d\theta$$
$$= W(\omega - \omega_0$$

Convolving with a Dirac delta function

So we see that:

$$\delta(\omega-\omega_0) * W(\omega) = W(\omega-\omega_0)$$

This is just like the behavior of impulses in the time domain:



Public domain,

https://commons.wikimedia.org/wiki/File:Convolution_of_two_pulses_with_impulse_response.svg

DTFT of a Windowed Cosine

So if:

$$\cos(\omega_0 n) \quad \leftrightarrow \quad \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0),$$

and

$$y[n]=x[n]w[n] \quad \leftrightarrow \quad Y(\omega)=\frac{1}{2\pi}X(\omega)*W(\omega),$$

then

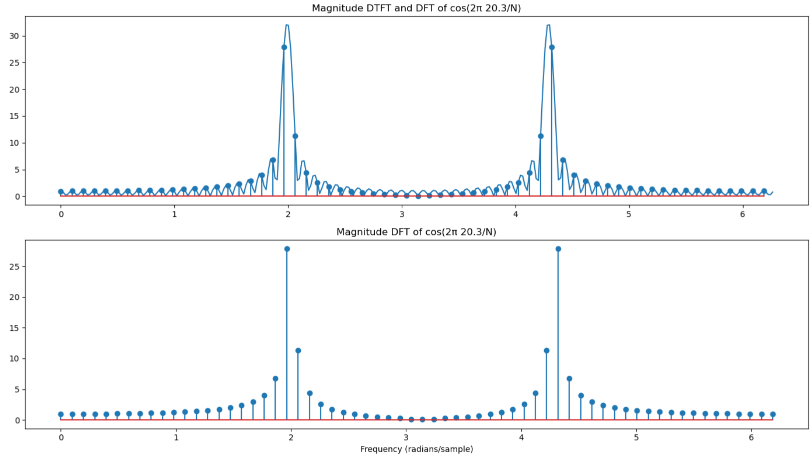
$$\cos(\omega_0 n)w[n] \quad \leftrightarrow \quad \left(\frac{1}{2}\delta(\omega-\omega_0)*W(\omega)+\frac{1}{2}\delta(\omega+\omega_0)*W(\omega)\right)$$

$$=\left(\frac{1}{2}W(\omega-\omega_0)+\frac{1}{2}W(\omega+\omega_0)\right)$$

DFT of a Cosine

So again, we discover that:

$$x[n] = \cos\left(\frac{2\pi 20.3}{N}n\right) w[n]$$



Outline

Summary

- DTFT of a complex exponential is a delta function:

$$e^{j\omega_0 n} \quad \leftrightarrow \quad 2\pi\delta(\omega-\omega_0)$$

- DTFT of a cosine is two delta functions:

$$\cos(\omega_0 n) \quad \leftrightarrow \quad \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$

- DTFT of a windowed cosine is frequency-shifted window functions:

$$\cos(\omega_0 n)w[n] \quad \leftrightarrow \quad \frac{1}{2}W(\omega-\omega_0)+\frac{1}{2}W(\omega+\omega_0)$$

Outline

Written Example

Consider the function

$$x[n] = A\cos(\omega_0 n + \theta)$$

What is $X(\omega)$?
How about $y[n] = w[n]x[n]$. What is $Y(\omega)$?