Review	FFT	Overlap-Add	Example

## Lecture 25: Overlap-Add

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## ECE 401: Signal and Image Analysis, Fall 2022

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Review	FFT	Overlap-Add	Example











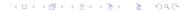
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Review: Circular	convolution		

Multiplying the DFT means **circular convolution** of the time-domain signals:

$$y[n] = h[n] \circledast x[n] \leftrightarrow Y[k] = H[k]X[k],$$

Circular convolution  $(h[n] \otimes x[n])$  is defined like this:

$$h[n] \circledast x[n] = \sum_{m=0}^{N-1} x[m]h[((n-m))_N] = \sum_{m=0}^{N-1} h[m]x[((n-m))_N]$$

Circular convolution is the same as linear convolution if and only if  $N \ge L + M - 1$ .

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Convolution is an  $O\{N^2\}$  operation: each of the N samples of y[n] is created by adding up N samples of x[m]h[n-m]:

$$y[n] = \sum_{m} x[m]h[n-m]$$

The way we've learned it so far, the DFT is **also** an  $O\{N^2\}$  operation: each of the *N* samples of X[k] is created by adding up *N* samples of  $x[n]e^{j\omega_k n}$ :

$$X[k] = \sum_{n} x[n] e^{-j\frac{2\pi kn}{N}}$$

However...

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The Fast Fourie	er Transform		

- The fast Fourier transform (FFT) is a clever divide-and-conquer algorithm that computes all of the N samples of X[k], from x[n], in only N log<sub>2</sub> N multiplications.
- It does this by computing all N of the X[k], all at once.
- Multiplications  $(x[n] \times w_{k,n})$ , for some coefficient  $w_{k,n}$  are grouped together, across different groups of k and n.
- On average, each of the N samples of X[k] can be computed using only log<sub>2</sub> N multiplications, for a total complexity of N log<sub>2</sub> N.

Consider filtering N = 1024 samples of audio (about 1/40 second) with a filter, h[n], that is 1024 samples long.

- Time-domain convolution requires  $1024 \times 1024 \approx 1,000,000$ multiplications. If a GPU does 40 billion multiplications/second, then it will take an hour of GPU time to apply this operation to a 1000-hour audio database.
- FFT requires 1024 × log<sub>2</sub> 1024 ≈ 10,000 multiplications. If a GPU does 40 billion multiplications/second, then it will take 36 seconds of GPU time to apply this operation to a 1000-hour audio database.

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How is it used?			

Suppose we have a 1025-sample h[n], and we want to filter a one-hour audio (144,000,000 samples). Divide the audio into frames, x[n], of length M = 1024, zero-pad to N = L + M - 1 = 2048, and take their FFTs.

- $H[k] = FFT\{h[n]\}$ : total cost is trivial, because we only need to do this once.
- $X[k] = FFT\{x[n]\}$ : total cost is  $N \log N$  per M samples.
- Y[k] = X[k]H[k]: total cost is N multiplications per M samples.

•  $y[n] = FFT^{-1}{Y[k]}$ : total cost is  $N \log N$  per M samples.

Grand total:  $N \times (2 \log_2 N + 1) = 2048 \times 23 = 47104$ multiplications per 1024 audio samples, or 46 multiplications per sample.

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How do w	e recombine the y	n]?	

- The main topic of today's lecture: how do we recombine the y[n]?
- Remember: each frame of x[n] was 1024 samples, but after zero-padding and convolution, each frame of y[n] is 2048 samples.

• How do we recombine them?

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Let's look more closely at what convolution is. Each sample of x[n] generates an impulse response. Those impulse responses are added together to make the output.

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First two lines show the first two frames (input on left, output on right). Last line shows the total input (left) and output (right).

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The Overlap-Add Algorithm

- Divide x[n] into frames
- Generate the output from each frame
- Overlap the outputs, and add them together

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The Overlap-Add Algorithm					

• Divide x[n] into frames (w[n] is a length-M rectangle).

 $x_t[n] = x[n + tM]w[n]$  $X_t[k] = FFT\{x_t[n]\}$ 

Generate the output from each frame

 $Y_t[k] = X_t[k]H[k]$  $y_t[n] = FFT^{-1}\{y_t[n]\}$ 

Overlap the outputs, and add them together

$$y[n] = \sum_{t} y_t[n - tM]$$

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Written Example					

- Suppose you have a billion samples of audio (about 6 hours' worth), and you want to convolve it with a 1025-sample lowpass filter. How many multiplications are required to do this using time-domain convolution? How many using overlap-add?
- Suppose that the audio is periodic, with a period of 1024 samples. Each period is 600 ones, followed by 424 zeros. Suppose that the filter is

$$h[n] = a^n, \ 0 \le n \le 1024$$

Use overlap-add (but with convolutions, not FFT) to find the first 2048 samples of y[n]

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