Lecture 24: Cicular Convolution

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis, Fall 2022

Outline

1. Review: DTFT and DFT
2. Sampled in Frequency $\leftrightarrow$ Periodic in Time
3. Circular Convolution
4. Zero-Padding
5. Summary

Review: DTFT

The DTFT (discrete time Fourier transform) of any signal is $X(\omega)$, given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n-n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$

Properties of the DTFT

Properties worth knowing include:

0. Periodicity: $X(\omega+2\pi) = X(\omega)$
1. Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$$

2. Time Shift: $x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
3. Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
4. Filtering is Convolution:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega) X(\omega)$$

Review: DFT

The DFT (discrete Fourier transform) of any signal is $X[k]$, given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$
$$x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F[k] = 1$$
$$g[n] = \delta\left[((n-n_0))_N\right] \leftrightarrow G[k] = e^{-j\frac{2\pi kn_0}{N}}$$

Properties of the DTFT

Properties worth knowing include:

0. Periodicity: $X[k+N] = X[k]$
1. Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z[k] = aX[k] + bY[k]$$

2. Circular Time Shift: $x\left[((n-n_0))_N\right] \leftrightarrow e^{-j\frac{2\pi k n_0}{N}} X(\omega)$
3. Frequency Shift: $e^{j\frac{2\pi k_0 n}{N}} x[n] \leftrightarrow X[k-k_0]$
4. Filtering is Circular Convolution:

$$y[n] = h[n] \circledast x[n] \leftrightarrow Y[k] = H[k]X[k],$$

Outline

Two different ways to think about the DFT

1. $x[n]$ is finite length; DFT is samples of DTFT

$$x[n] = 0, n < 0 \text{ or } n \geq N \quad \leftrightarrow \quad X[k] = X(\omega)\big|_{\omega=\frac{2\pi k}{N}}$$

2. $x[n]$ is periodic; DFT is scaled version of Fourier series

$$x[n] = x[n+N] \quad \leftrightarrow \quad X[k] = NX_k$$

1. $x[n]$ finite length, DFT is samples of DTFT

If $x[n]$ is nonzero only for $0 \leq n \leq N-1$, then

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n},$$

and

$$X[k] = X(\omega)\big|_{\omega=\frac{2\pi k}{N}}$$

2. $x[n]$ periodic, $X[k] = NX_k$

If $x[n] = x[n+N]$, then its Fourier series is

$$X_k = \frac{1}{N}\sum_{n=1}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}},$$

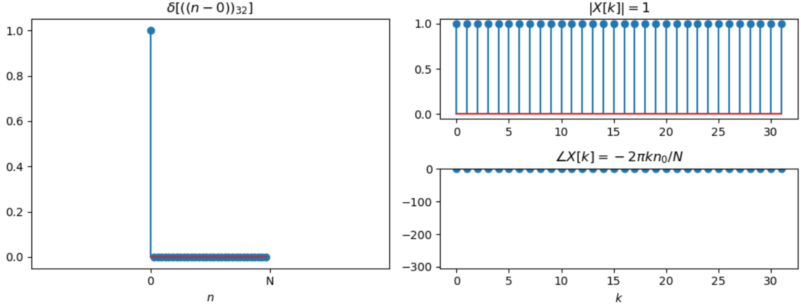
and its DFT is

$$X[k] = \sum_{n=1}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}$$

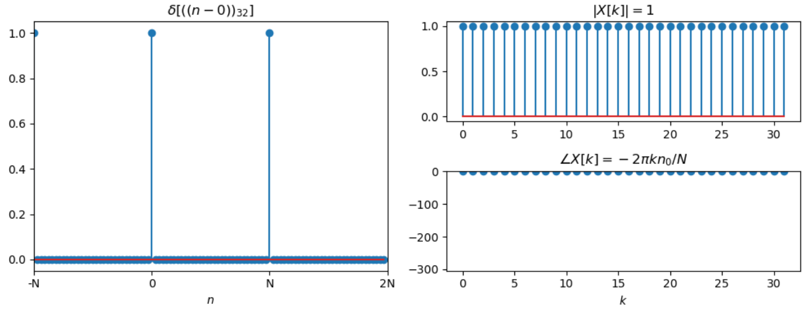
Delayed impulse wraps around

$$\delta\left[((n-n_0))_N\right] \leftrightarrow e^{-j\frac{2\pi k n_0}{N}}$$



Delayed impulse is really periodic impulse

$$\delta\left[((n-n_0))_N\right] \leftrightarrow e^{-j\frac{2\pi k n_0}{N}}$$



Principal Phase

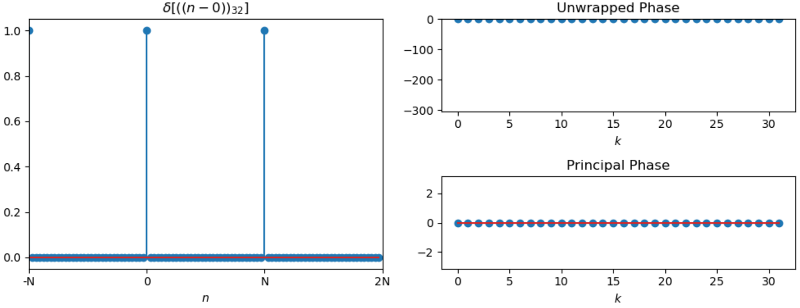
- Something weird going on: how can the phase keep getting bigger and bigger, but the signal wraps around?
- It's because the phase wraps around too!

$$\angle X[k] = -\omega_k(N+n) = -\omega_k n, \quad \text{because } \omega_k = \frac{2\pi k}{N}$$

- **Principal phase =** add $\pm 2\pi$ to the phase, as necessary, so that $-\pi < \angle X[k] \leq \pi$
- **Unwrapped phase =** let the phase be as large as necessary so that it is plotted as a smooth function of ω

Unwrapped phase vs. Principal phase

$$\delta\left[((n-n_0))_N\right] \leftrightarrow e^{-j\frac{2\pi k n_0}{N}}$$



Summary: Two different ways to think about the DFT

1. $x[n]$ is finite length; DFT is samples of DTFT

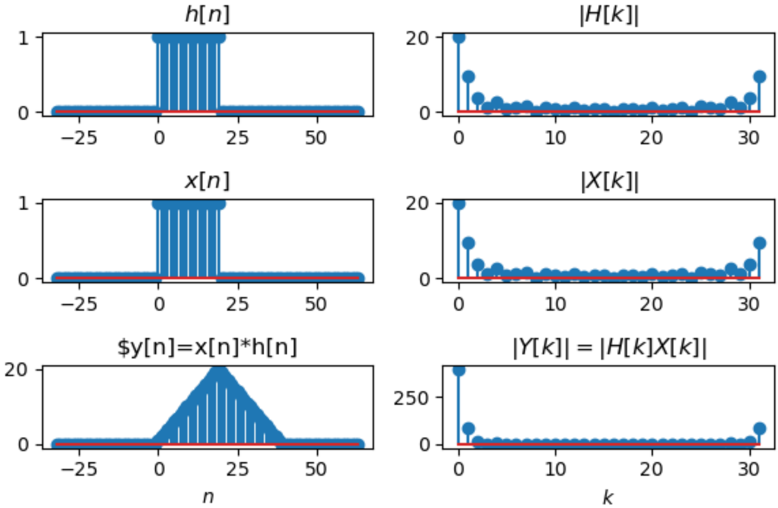
$$x[n] = 0, n < 0 \text{ or } n \geq N \quad \leftrightarrow \quad X[k] = X(\omega)\big|_{\omega=\frac{2\pi k}{N}}$$

2. $x[n]$ is periodic; DFT is scaled version of Fourier series

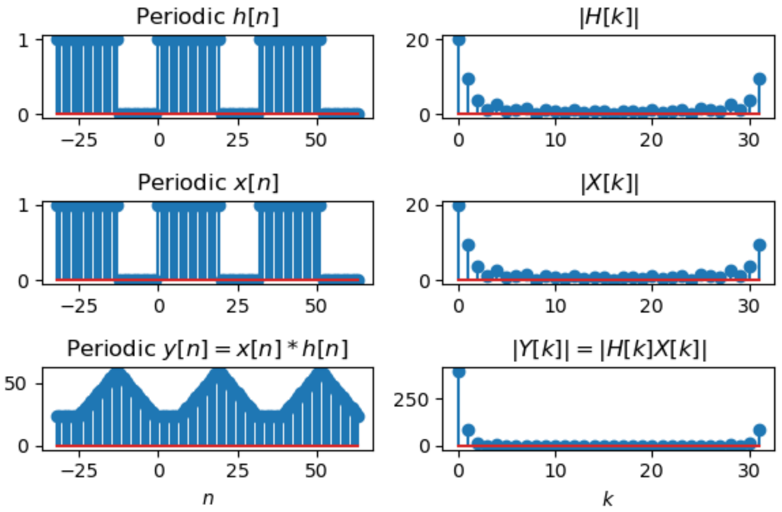
$$x[n] = x[n+N] \quad \leftrightarrow \quad X[k] = NX_k$$

Outline

Multiplying two DFTs: what we think we're doing



Multiplying two DFTs: what we're actually doing



Circular convolution

Suppose $Y[k] = H[k]X[k]$, then

$$
\begin{aligned}
y[n] &= \frac{1}{N}\sum_{k=0}^{N-1} H[k]X[k]e^{j\frac{2\pi kn}{N}} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} H[k]\left(\sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi km}{N}}\right)e^{j\frac{2\pi kn}{N}} \\
&= \sum_{m=0}^{N-1} x[m]\left(\frac{1}{N}\sum_{k=0}^{N-1} H[k]e^{-j\frac{2\pi k(n-m)}{N}}\right) \\
&= \sum_{m=0}^{N-1} x[m]h\left[((n-m))_N\right]
\end{aligned}
$$

The last line is because $\frac{2\pi k(n-m)}{N} = \frac{2\pi k((n-m))_N}{N}$.

Circular convolution

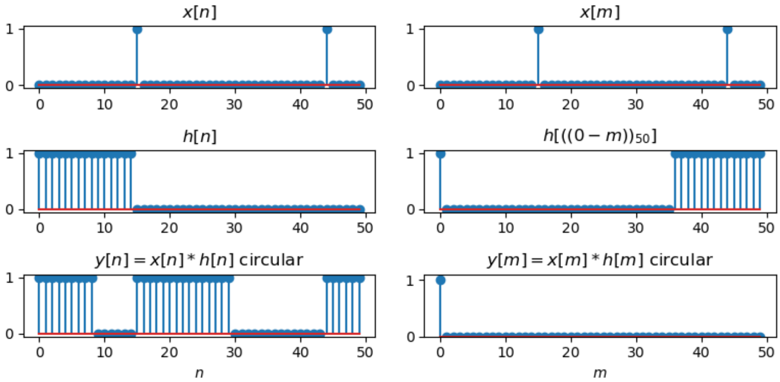
Multiplying the DFT means **circular convolution** of the time-domain signals:

$$y[n] = h[n] \circledast x[n] \leftrightarrow Y[k] = H[k]X[k],$$

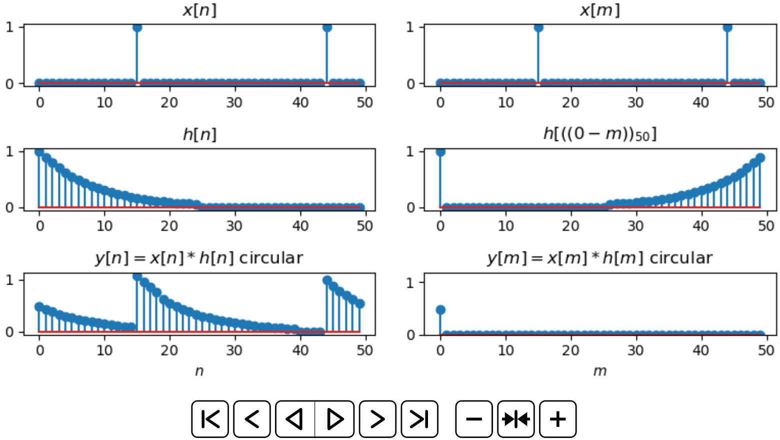
Circular convolution ($h[n] \circledast x[n]$) is defined like this:

$$h[n] \circledast x[n] = \sum_{m=0}^{N-1} x[m] h\left[((n-m))_N\right] = \sum_{m=0}^{N-1} h[m] x\left[((n-m))_N\right]$$

Circular convolution example



Circular convolution example



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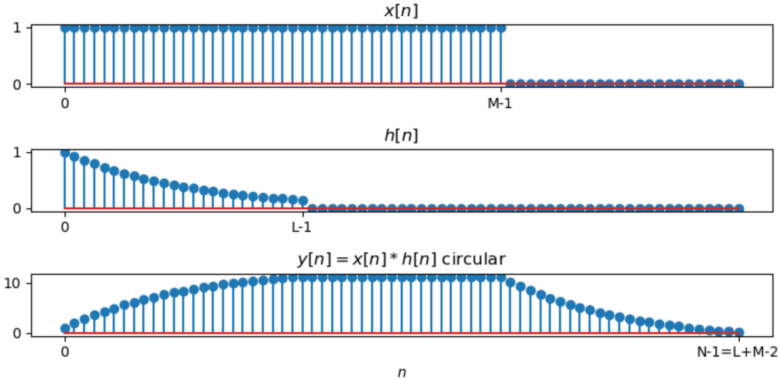
How long is $h[n] * x[n]$?

If $x[n]$ is M samples long, and $h[n]$ is L samples long, then their linear convolution, $y[n] = x[n] * h[n]$, is $M + L - 1$ samples long.



Zero-padding turns circular convolution into linear convolution

How it works:

- $h[n]$ is length-L
- $x[n]$ is length-M
- As long as they are both zero-padded to length $N \geq L + M - 1$, then
- $y[n] = h[n] \circledast x[n]$ is the same as $h[n] * x[n]$.

Zero-padding turns circular convolution into linear convolution

Why it works: Either...

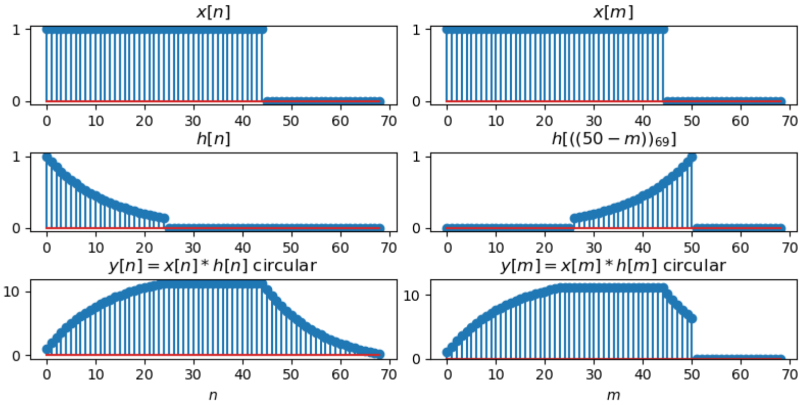
- $n-m$ is a positive number, between 0 and $N-1$. Then $((n-m))_N = n-m$, and therefore

$$x[m]h\left[((n-m))_N\right] = x[m]h[n-m]$$

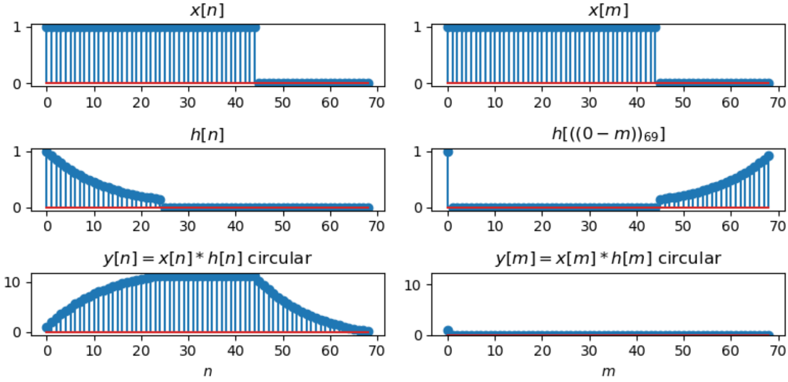
- $n-m$ is a negative number, between 0 and $-(L-1)$. Then $((n-m))_N = N+n-m \geq N-(L-1) > M-1$, so

$$x[m]h\left[((n-m))_N\right] = 0$$

Case #1: $n - m$ is positive, so circular convolution is the same as linear convolution



Case #2: $n - m$ is negative, so it wraps around, but N is long enough so that the wrapped part of $h[((n-m))_N]$ doesn't overlap with $x[m]$



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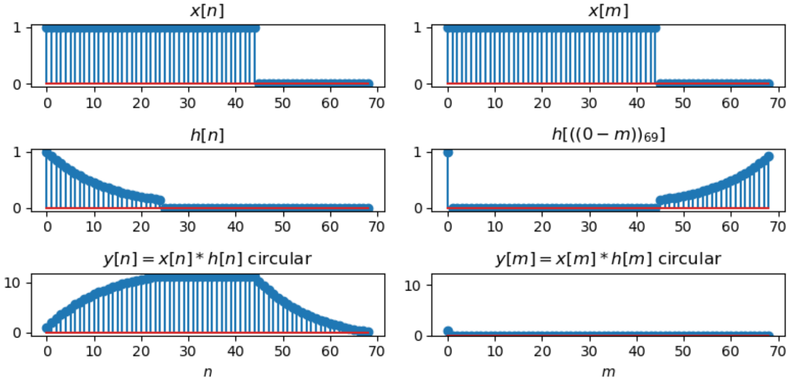

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Circular convolution is the same as linear convolution if and only if $N \geq L + M - 1$.