Lecture 17: Ideal Filters

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ECE 401: Signal and Image Analysis, Fall 2022
1. Review: DTFT
2. Ideal Lowpass Filter
3. Ideal Highpass Filter
4. Ideal Bandpass Filter
5. Summary
6. Written Example
Outline

1. Review: DTFT
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The DTFT (discrete time Fourier transform) of any signal is $X(\omega)$, given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n - n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$
Properties of the DTFT

Properties worth knowing include:

1. **Periodicity:** \( X(\omega + 2\pi) = X(\omega) \)
2. **Linearity:**
   \[
   z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)
   \]
3. **Time Shift:** \( x[n - n_0] \leftrightarrow e^{-j\omega n_0}X(\omega) \)
4. **Frequency Shift:** \( e^{j\omega_0 n}x[n] \leftrightarrow X(\omega - \omega_0) \)
5. **Filtering is Convolution:**
   \[
   y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)
   \]
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What is “Ideal”?

The definition of “ideal” depends on your application. Let’s start with the task of lowpass filtering. Let’s define an ideal lowpass filter, \( Y(\omega) = H_{LP}(\omega)X(\omega) \), as follows:

\[
Y(\omega) = \begin{cases} 
  X(\omega) & |\omega| \leq \omega_c, \\
  0 & \text{otherwise},
\end{cases}
\]

where \( \omega_c \) is some cutoff frequency that we choose. For example, to de-noise a speech signal we might choose \( \omega_c = 2\pi 2400 / F_s \), because most speech energy is below 2400Hz. This definition gives:

\[
H_{LP}(\omega) = \begin{cases} 
  1 & |\omega| \leq \omega_c \\
  0 & \text{otherwise}
\end{cases}
\]
Ideal Lowpass Filter

$|X(\omega)|$

$H_{LP}(\omega)$

$|Y(\omega)| = H_{LP}(\omega)|X(\omega)|$
How can we implement an ideal LPF?

1. Use `np.fft.fft` to find $X[k]$, set $Y[k] = X[k]$ only for $\frac{2\pi k}{N} < \omega_c$, then use `np.fft.ifft` to convert back into the time domain?
   - It sounds easy, but...
   - `np.fft.fft` is finite length, whereas the DTFT is infinite length. Truncation to finite length causes artifacts.

2. Use pencil and paper to inverse DTFT $H_{LP}(\omega)$ to $h_{LP}[n]$, then use `np.convolve` to convolve $h_{LP}[n]$ with $x[n]$.
   - It sounds more difficult.
   - But actually, we only need to find $h_{LP}[n]$ once, and then we’ll be able to use the same formula for ever afterward.
   - This method turns out to be both easier and more effective in practice.
Inverse DTFT of $H_{LP}(\omega)$

The ideal LPF is

$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

The inverse DTFT is

$$h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(\omega)e^{j\omega n} d\omega$$

Combining those two equations gives

$$h_{LP}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$
Solving the integral

The ideal LPF is

\[ h_{LP}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \left( \frac{1}{jn} \right) \left[ e^{j\omega n} \right]_{-\omega_c}^{\omega_c} \]

\[ = \frac{1}{2\pi} \left( \frac{1}{jn} \right) (2j \sin(\omega_c n)) \]

\[ = \frac{\sin(\omega_c n)}{\pi n} \]

\[ = \left( \frac{\omega_c}{\pi} \right) \text{sinc}(\omega_c n) \]
\[ h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n} \]

- \( \frac{\sin(\omega_c n)}{\pi n} \) is undefined when \( n = 0 \)
- \( \lim_{n \to 0} \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi} \)
- So let’s define \( h_{LP}[0] = \frac{\omega_c}{\pi} \).
\[ h_{LP}[n] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \]
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Ideal Highpass Filter

An ideal high-pass filter passes all frequencies above $\omega_c$:

$$H_{HP}(\omega) = \begin{cases} 1 & |\omega| > \omega_c \\ 0 & \text{otherwise} \end{cases}$$
Ideal Highpass Filter

...except for one problem: aliasing.
The highest frequency, in discrete time, is $\omega = \pi$. Frequencies that seem higher, like $\omega = 1.1\pi$, are actually lower. This phenomenon is called “aliasing.”
Ideal Highpass Filter

Here’s how an ideal HPF looks if we only plot from $-\pi \leq \omega \leq \pi$:
Here's how an ideal HPF looks if we plot from $-2\pi \leq \omega \leq 2\pi$:
Ideal Highpass Filter

Here’s how an ideal HPF looks if we plot from $-3\pi \leq \omega \leq 3\pi$:
Redefining "Lowpass" and "Highpass"

Let’s redefine “lowpass” and “highpass.” The ideal LPF is

$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c, \\ 0 & \omega_c < |\omega| \leq \pi. \end{cases}$$

The ideal HPF is

$$H_{HP}(\omega) = \begin{cases} 0 & |\omega| < \omega_c, \\ 1 & \omega_c \leq |\omega| \leq \pi. \end{cases}$$

Both of them are periodic with period $2\pi$. 
The easiest way to find $h_{HP}[n]$ is to use linearity:

$$H_{HP}(\omega) = 1 - H_{LP}(\omega)$$

Therefore:

$$h_{HP}[n] = \delta[n] - h_{LP}[n]$$

$$= \delta[n] - \frac{\omega_c}{\pi} \text{sinc}(\omega_c n)$$
\[ h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \]
Comparing highpass and lowpass filters

$H_{LP}(\omega)$, cutoff=$\pi/4$

$H_{LP}(\omega)$, cutoff=$\pi/2$

$H_{LP}(\omega)$, cutoff=$3\pi/4$

$h_{LP}[n]$, cutoff=$\pi/4$

$h_{LP}[n]$, cutoff=$\pi/2$

$h_{LP}[n]$, cutoff=$3\pi/4$
$$h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi} \text{sinc}(\omega_c n)$$
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Ideal Bandpass Filter

An ideal band-pass filter passes all frequencies between $\omega_1$ and $\omega_2$:

$$H_{BP}(\omega) = \begin{cases} 
1 & \omega_1 \leq |\omega| \leq \omega_2 \\
0 & \text{otherwise}
\end{cases}$$

(and, of course, it’s also periodic with period $2\pi$).
The easiest way to find \( h_{BP}[n] \) is to use linearity:

\[
H_{BP}(\omega) = H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega)
\]

Therefore:

\[
h_{BP}[n] = \frac{\omega_2}{\pi} \text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi} \text{sinc}(\omega_1 n)
\]
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Summary: Ideal Filters

- Ideal Lowpass Filter:
  \[ H_{LP}(\omega) = \begin{cases} 
  1 & |\omega| \leq \omega_c, \\
  0 & \omega_c < |\omega| \leq \pi.
\end{cases} \quad \leftrightarrow \quad h_{LP}[m] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \]

- Ideal Highpass Filter:
  \[ H_{HP}(\omega) = 1 - H_{LP}(\omega) \quad \leftrightarrow \quad h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \]

- Ideal Bandpass Filter:
  \[ H_{BP}(\omega) = H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega) \]
  \[ \leftrightarrow h_{BP}[n] = \frac{\omega_2}{\pi} \text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi} \text{sinc}(\omega_1 n) \]
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Suppose you have an image with a sharp boundary, between black and white, at the location $n = 0$. This is well modeled by setting $x[n]$ equal to the unit step function:

$$x[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Use graphical convolution to convolve $x[n]$ with an ideal LPF. You don’t need to find the exact values of $y[n]$, but sketch things like: how wide is the ramp between light and dark? How frequent are the ripples on either side of the ramp?