Lecture 13: Frequency Response

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1. Review: Convolution and Fourier Series

2. Frequency Response

3. Example: First Difference

4. Superposition and the Frequency Response

5. Example: First Difference

6. Written Examples

7. Summary
Outline

1. Review: Convolution and Fourier Series
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What is Signal Processing, Really?

- When we process a signal, usually, we’re trying to enhance the meaningful part, and reduce the noise.
- **Spectrum** helps us to understand which part is meaningful, and which part is noise.
- **Convolution** (a.k.a. filtering) is the tool we use to perform the enhancement.
- **Frequency Response** of a filter tells us exactly which frequencies it will enhance, and which it will reduce.
A **convolution** is exactly the same thing as a **weighted local average**. We give it a special name, because we will use it very often. It’s defined as:

\[ y[n] = \sum_{m} h[m]f[n - m] = \sum_{m} h[n - m]f[m] \]

We use the symbol \( \ast \) to mean “convolution:”

\[ y[n] = h[n] \ast f[n] = \sum_{m} h[m]f[n - m] = \sum_{m} h[n - m]f[m] \]
Review: Spectrum

The **spectrum** of \( x(t) \) is the set of frequencies, and their associated phasors,

\[
\text{Spectrum} \left( x(t) \right) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}
\]

such that

\[
x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}
\]
One reason the spectrum is useful is that any periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t + T_0) = x(t)$$

can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kF_0 t}$$

which is a special case of the spectrum for periodic signals: $f_k = kF_0$, and $a_k = X_k$, and

$$F_0 = \frac{1}{T_0}$$
**Fourier Series Analysis** (finding the spectrum, given the waveform):

\[ X_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0} \, dt \]

**Fourier Series Synthesis** (finding the waveform, given the spectrum):

\[ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} \]
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Frequency Response

The **frequency response**, $H(\omega)$, of a filter $h[n]$, is its output in response to a pure tone, expressed as a function of the frequency of the tone.
Calculating the Frequency Response

- Output of the filter:
  \[ y[n] = h[n] \ast x[n] \]
  \[ = \sum_{m} h[m]x[n - m] \]

- In response to a pure tone:
  \[ x[n] = e^{j\omega n} \]
Calculating the Frequency Response

Output of the filter in response to a pure tone:

\[ y[n] = \sum_{m} h[m]x[n - m] \]
\[ = \sum_{m} h[m]e^{j\omega(n-m)} \]
\[ = e^{j\omega n} \left( \sum_{m} h[m]e^{-j\omega m} \right) \]

Notice that the part inside the parentheses is not a function of \( n \). It is not a function of \( m \), because the \( m \) gets summed over. It is only a function of \( \omega \). It is called the frequency response of the filter, \( H(\omega) \).
Frequency Response

When the input to a filter is a pure tone,

\[ x[n] = e^{j\omega n}, \]

then its output is the same pure tone, scaled and phase shifted by a complex number called the **frequency response** \( H(\omega) \):

\[ y[n] = H(\omega)e^{j\omega n} \]

The frequency response is related to the impulse response as

\[ H(\omega) = \sum_{m} h[m]e^{-j\omega m} \]
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Example: First Difference

Remember that taking the difference between samples can be written as a convolution:

\[ y[n] = x[n] - x[n - 1] = h[n] \ast x[n], \]

where

\[ h[n] = \begin{cases} 
1 & n = 0 \\
-1 & n = 1 \\
0 & \text{otherwise}
\end{cases} \]
Example: First Difference

Suppose that the input is a pure tone:

\[ x[n] = e^{j\omega n} \]

Then the output will be

\[
\begin{align*}
y[n] &= x[n] - x[n - 1] \\
   &= e^{j\omega n} - e^{j\omega(n-1)} \\
   &= H(\omega)e^{j\omega n},
\end{align*}
\]

where

\[ H(\omega) = 1 - e^{-j\omega} \]
Example: First Difference

So we have some pure-tone input,

\[ x[n] = e^{j\omega n} \]

\[ y[n] = x[n] - x[n - 1] \]

\[ y[n] = H(\omega) e^{j\omega n} \]
Example: First Difference

- How much is the scaling?
- How much is the phase shift?

Let’s find out.

\[ H(\omega) = 1 - e^{-j\omega} \]

\[ = e^{-j\frac{\omega}{2}} \left( e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \right) \]

\[ = e^{-j\frac{\omega}{2}} \left( 2j \sin \left( \frac{\omega}{2} \right) \right) \]

\[ = \left( 2 \sin \left( \frac{\omega}{2} \right) \right) \left( e^{j\left(\frac{\pi-\omega}{2}\right)} \right) \]

So the magnitude and phase response are:

\[ |H(\omega)| = 2 \sin \left( \frac{\omega}{2} \right) \]

\[ \angle H(\omega) = \frac{\pi - \omega}{2} \]
First Difference: Magnitude Response

\[ |H(\omega)| = 2 \sin \left( \frac{\omega}{2} \right) \]
First Difference Filter at $\omega = 0$

Suppose we put in the signal $x[n] = e^{j\omega n}$, but at the frequency $\omega = 0$. At that frequency, $x[n] = 1$. So

$$y[n] = x[n] - x[n - 1] = 0$$
First Difference Filter at $\omega = \pi$

Frequency $\omega = \pi$ means the input is $(-1)^n$:

$$x[n] = e^{j\pi n} = (-1)^n = \begin{cases} 1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{cases}$$

So

$$y[n] = x[n] - x[n-1] = 2x[n]$$
First Difference Filter at $\omega = \frac{\pi}{2}$

Frequency $\omega = \frac{\pi}{2}$ means the input is $j^n$:

$$x[n] = e^{j \frac{\pi n}{2}} = j^n$$

The frequency response is:

$$G \left( \frac{\pi}{2} \right) = 1 - e^{-j \frac{\pi}{2}} = 1 - \left( \frac{1}{j} \right),$$

The output is

$$y[n] = x[n] - x[n-1] = j^n - j^{n-1} = \left( 1 - \frac{1}{j} \right) j^n$$
Superposition and the Frequency Response

The frequency response obeys the principle of **superposition**, meaning that if the input is the sum of two pure tones:

\[ x[n] = e^{j\omega_1 n} + e^{j\omega_2 n}, \]

then the output is the sum of the same two tones, each scaled by the corresponding frequency response:

\[ y[n] = H(\omega_1)e^{j\omega_1 n} + H(\omega_2)e^{j\omega_2 n} \]
Response to a Cosine

There are no complex exponentials in the real world. Instead, we’d like to know the output in response to a cosine input. Fortunately, a cosine is the sum of two complex exponentials:

\[ x[n] = \cos(\omega n) = \frac{1}{2} e^{j\omega n} + \frac{1}{2} e^{-j\omega n}, \]

therefore,

\[ y[n] = \frac{H(\omega)}{2} e^{j\omega n} + \frac{H(-\omega)}{2} e^{-j\omega n} \]
Response to a Cosine

What is $H(-\omega)$? Remember the definition:

$$H(\omega) = \sum_{m} h[m] e^{-j\omega m}$$

Replacing every $\omega$ with a $-\omega$ gives:

$$H(-\omega) = \sum_{m} h[m] e^{j\omega m}.$$  

Notice that $h[m]$ is real-valued, so the only complex number on the RHS is $e^{j\omega m}$. But

$$e^{j\omega m} = (e^{-j\omega m})^*$$

so

$$H(-\omega) = H^*(\omega)$$
Response to a Cosine

\[ y[n] = \frac{H(\omega)}{2} e^{j\omega n} + \frac{H^*(\omega)}{2} e^{-j\omega n} \]

\[ = \frac{|H(\omega)|}{2} e^{j\angle H(\omega)} e^{j\omega n} + \frac{|H(\omega)|}{2} e^{-j\angle H(\omega)} e^{-j\omega n} \]

\[ = |H(\omega)| \cos (\omega n + \angle H(\omega)) \]
### Response to a Cosine

If

\[ x[n] = \cos(\omega n) \]

... then ...

\[ y[n] = |H(\omega)| \cos (\omega n + \angle H(\omega)) \]

### Magnitude and Phase Responses

- The **Magnitude Response** \( |H(\omega)| \) tells you by how much a pure tone at \( \omega \) will be scaled.
- The **Phase Response** \( \angle H(\omega) \) tells you by how much a pure tone at \( \omega \) will be advanced in phase.
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Example: First Difference

Remember that the first difference, $y[n] = x[n] - x[n-1]$, is supposed to sort of approximate a derivative operator:

$$y(t) \approx \frac{d}{dt} x(t)$$

If the input is a cosine, what is the output?

$$\frac{d}{dt} \cos(\omega t) = -\omega \sin(\omega t) = \omega \cos\left(\omega t + \frac{\pi}{2}\right)$$

Does the first-difference operator behave the same way (multiply by a magnitude of $|H(\omega)| = \omega$, phase shift by $+\frac{\pi}{2}$ radians so that cosine turns into negative sine)?
Example: First Difference

Frequency response of the first difference filter is

\[ H(\omega) = 1 - e^{-j\omega} \]

Let's try to convert it to polar form, so we can find its magnitude and phase:

\[ H(\omega) = e^{-j\frac{\omega}{2}} \left( e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \right) \]
\[ = e^{-j\frac{\omega}{2}} \left( 2j \sin \left( \frac{\omega}{2} \right) \right) \]
\[ = \left( 2 \sin \left( \frac{\omega}{2} \right) \right) \left( e^{j\left( \frac{\pi - \omega}{2} \right)} \right) \]

So the magnitude and phase response are:

\[ |H(\omega)| = 2 \sin \left( \frac{\omega}{2} \right) \]
\[ \angle H(\omega) = \frac{\pi - \omega}{2} \]
First Difference: Magnitude Response

Taking the derivative of a cosine scales it by $\omega$. The first-difference filter scales it by $|H(\omega)| = 2 \sin(\omega/2)$, which is almost the same, but not quite:
First Difference: Phase Response

Taking the derivative of a cosine shifts it, in phase, by $\pm \frac{\pi}{2}$ radians, so that the cosine turns into a negative sine. The first-difference filter shifts it by $\angle H(\omega) = \frac{\pi - \omega}{2}$, which is the same at very low frequencies, but very different at high frequencies.
Putting it all together, if the input is $x[n] = \cos(\omega n)$, the output is

$$y[n] = |H(\omega)| \cos (\omega n + \angle H(\omega)) = 2 \sin \left( \frac{\omega}{2} \right) \cos \left( \omega n + \frac{\pi - \omega}{2} \right)$$
First Difference: All Together

\[ y[n] = 2 \sin \left( \frac{\omega}{2} \right) \cos \left( \omega n + \frac{\pi - \omega}{2} \right) \]

At very low frequencies, the output is almost \(-\sin(\omega n)\), but with very low amplitude:
First Difference: All Together

\[ y[n] = 2 \sin \left( \frac{\omega}{2} \right) \cos \left( \omega n + \frac{\pi - \omega}{2} \right) \]

At intermediate frequencies, the phase shift between the input and output is reduced:

\[ x[n] = \cos(\omega n) \text{ and } y[n] = x[n] - x[n-1], \omega = 1.57 \]
First Difference: All Together

\[ y[n] = 2 \sin \left( \frac{\omega}{2} \right) \cos \left( \omega n + \frac{\pi - \omega}{2} \right) \]

At very high frequencies, the phase shift between input and output is eliminated – the output is a cosine, just like the input:
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Consider the following system:

\[ y[n] = x[n - n_0] \]

This can be written as a convolution, with impulse response

\[ h[n] = \delta[n - n_0] \]

What is \( H(\omega) \)? Using \( H(\omega) \), can you find the output of this system if \( x[n] = \cos(\omega n) \)?
Tones in $\rightarrow$ Tones out

\[ x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n} \]
\[ x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)| \cos(\omega n + \angle H(\omega)) \]
\[ x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|H(\omega)| \cos(\omega n + \theta + \angle H(\omega)) \]

where the **Frequency Response** is given by

\[ H(\omega) = \sum_{m} h[m]e^{-j\omega m} \]