Lecture 12: Impulse Response

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis, Fall 2021
1. Review: Linearity and Shift Invariance
2. Convolution
3. Written Example
4. Summary
Outline

1. Review: Linearity and Shift Invariance
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4. Summary
A **system** is anything that takes one signal as input, and generates another signal as output. We can write

\[ x[n] \xrightarrow{\mathcal{H}} y[n] \]

which means

\[ x[n] \xrightarrow{} \mathcal{H} \xrightarrow{} y[n] \]
A system is **linear** if and only if, for any two inputs $x_1[n]$ and $x_2[n]$ that produce outputs $y_1[n]$ and $y_2[n]$,

$$x[n] = x_1[n] + x_2[n] \xrightarrow{\mathcal{H}} y[n] = y_1[n] + y_2[n]$$

A system is **shift-invariant** if and only if, for any input $x_1[n]$ that produces output $y_1[n]$,

$$x[n] = x_1[n - n_0] \xrightarrow{\mathcal{H}} y[n] = y_1[n - n_0]$$
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We care about linearity and shift-invariance because of the following remarkable result:

Let $\mathcal{H}$ be any system,

$$x[n] \xrightarrow{\mathcal{H}} y[n]$$

If $\mathcal{H}$ is linear and shift-invariant, then whatever processes it performs can be equivalently replaced by a convolution:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] x[n - m]$$
Impulse Response

\[ y[n] = \sum_{m=-\infty}^{\infty} h[m] \times [n - m] \]

The weights \( h[m] \) are called the “impulse response” of the system. We can measure them, in the real world, by putting the following signal into the system:

\[ \delta[n] = \begin{cases} 
1 & n = 0 \\
0 & \text{otherwise} 
\end{cases} \]

and measuring the response:

\[ \delta[n] \xrightarrow{H} h[n] \]
Convolution: Proof

1. $h[n]$ is the impulse response.

$$\delta[n] \xrightarrow{H} h[n]$$

2. The system is **shift-invariant**, therefore

$$\delta[n - m] \xrightarrow{H} h[n - m]$$

3. The system is **linear**, therefore scaling the input by a constant results in scaling the output by the same constant:

$$x[m] \delta[n - m] \xrightarrow{H} x[m] h[n - m]$$

4. The system is **linear**, therefore adding input signals results in adding the output signals:

$$\sum_{m=-\infty}^{\infty} x[m] \delta[n - m] \xrightarrow{H} \sum_{m=-\infty}^{\infty} x[m] h[n - m]$$
Convolution: Proof (in Words)

- The input signal, $x[n]$, is just a bunch of samples.
- Each one of those samples is a scaled impulse, so each one of them produces a scaled impulse response at the output.
- Convolution = add together those scaled impulse responses.
Convolution: Proof (in Pictures)

\[ x[0] \delta[n-0] \]

\[ x[1] \delta[n-1] \]

\[ x[2] \delta[n-2] \]

\[ x[3] \delta[n-3] \]

\[ x[n] = \sum_{m} x[m] \delta[n-m] \]

\[ y[n] = \sum_{m} x[m] h[n-m] \]
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Consider a system that computes the summation of all of its inputs:

\[ y[n] = \sum_{m=-\infty}^{n} x[m] \]

What is the impulse response of this system? Show that this system can be implemented using \( y[n] = h[n] \ast x[n] \) for an appropriate \( h[n] \).
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A system is **linear** if and only if, for any two inputs $x_1[n]$ and $x_2[n]$ that produce outputs $y_1[n]$ and $y_2[n]$,

$$x[n] = x_1[n] + x_2[n] \xrightarrow{\mathcal{H}} y[n] = y_1[n] + y_2[n]$$

A system is **shift-invariant** if and only if, for any input $x_1[n]$ that produces output $y_1[n]$,

$$x[n] = x_1[n - n_0] \xrightarrow{\mathcal{H}} y[n] = y_1[n - n_0]$$

If a system is **linear and shift-invariant** (LSI), then it can be implemented using convolution:

$$y[n] = h[n] \ast x[n]$$

where $h[n]$ is the impulse response:

$$\delta[n] \xrightarrow{\mathcal{H}} h[n]$$