Lecture 11: Linearity and Shift-Invariance

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1. Systems
2. Linearity
3. Shift Invariance
4. Written Example
5. Summary
Outline

1. Systems
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A **system** is anything that takes one signal as input, and generates another signal as output. We can write

\[ x[n] \xrightarrow{H} y[n] \]

which means

\[ x[n] \longrightarrow \mathcal{H} \longrightarrow y[n] \]
Example: Averager

For example, a weighted local averager is a system. Let’s call it system $A$.

$$x[n] \xrightarrow{A} y[n] = \sum_{m=0}^{6} g[m] x[n - m]$$
A time-shift is a system. Let’s call it system $\mathcal{T}$.

$$x[n] \xrightarrow{\mathcal{T}} y[n] = x[n - 1]$$
Example: Square

If you calculate the square of a signal, that’s also a system. Let’s call it system $S$:

$$x[n] \xrightarrow{S} y[n] = x^2[n]$$
Example: Add a Constant

If you add a constant to a signal, that’s also a system. Let’s call it system $C$:

$$x[n] \xrightarrow{C} y[n] = x[n] + 1$$
Example: Window

If you chop off all elements of a signal that are before time 0 or after time $N - 1$ (for example, because you want to put it into an image), that is a system:

$$x[n] \xrightarrow{\mathcal{W}} y[n] = \begin{cases} x[n] & 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases}$$
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A system is **linear** if these two algorithms compute the same thing:
A system $\mathcal{H}$ is said to be **linear** if and only if, for any $x_1[n]$ and $x_2[n]$,

$$x_1[n] \xrightarrow{\mathcal{H}} y_1[n]$$

$$x_2[n] \xrightarrow{\mathcal{H}} y_2[n]$$

implies that

$$x[n] = x_1[n] + x_2[n] \xrightarrow{\mathcal{H}} y[n] = y_1[n] + y_2[n]$$

In words: a system is **linear** if and only if, for every pair of inputs $x_1[n]$ and $x_2[n]$, (1) adding the inputs and then passing them through the system gives exactly the same effect as (2) passing both inputs through the system, and **then** adding them.
Notice, a special case of linearity is the case when $x_1[n] = x_2[n]$:

$$x_1[n] \xrightarrow{H} y_1[n]$$

$$x_1[n] \xrightarrow{H} y_1[n]$$

implies that

$$x[n] = 2x_1[n] \xrightarrow{H} y[n] = 2y_1[n]$$

So if a system is linear, then **scaling the input** also **scales the output**.
Example: Averager

Let’s try it with the weighted averager.

\[
x_1[n] \xrightarrow{A} y_1[n] = \sum_{m=0}^{6} g[m]x_1[n - m]
\]

\[
x_2[n] \xrightarrow{A} y_2[n] = \sum_{m=0}^{6} g[m]x_2[n - m]
\]

Then:

\[
x[n] = x_1[n] + x_2[n] = \sum_{m=0}^{6} g[m] (x_1[n - m] + x_2[n - m])
\]

\[
= \left( \sum_{m=0}^{6} g[m]x_1[n - m] \right) + \left( \sum_{m=0}^{6} g[m]x_2[n - m] \right)
\]

\[
= y_1[n] + y_2[n]
\]

… so a weighted averager is a linear system.
A squarer is just obviously nonlinear, right? Let’s see if that’s true:

\[
x_1[n] \xrightarrow{S} y_1[n] = x_1^2[n]
\]

\[
x_2[n] \xrightarrow{S} y_2[n] = x_2^2[n]
\]

Then:

\[
x[n] = x_1[n] + x_2[n] \xrightarrow{A} y[n] = x^2[n]
\]

\[
= (x_1[n] + x_2[n])^2
\]

\[
= x_1^2[n] + 2x_1[n]x_2[n] + x_2^2[n]
\]

\[
\neq y_1[n] + y_2[n]
\]

...so a squarer is a **nonlinear system**.
Example: Add a Constant

This one is tricky. Adding a constant seems like it ought to be linear, but it’s actually **nonlinear**. Adding a constant is what’s called an **affine** system, which is not necessarily linear.

\[
\begin{align*}
x_1[n] &\overset{c}{\rightarrow} y_1[n] = x_1[n] + 1 \\
x_2[n] &\overset{c}{\rightarrow} y_2[n] = x_2[n] + 1
\end{align*}
\]

Then:

\[
\begin{align*}
x[n] &= x_1[n] + x_2[n] \overset{A}{\rightarrow} y[n] = x[n] + 1 \\
&= x_1[n] + x_2[n] + 1 \\
&\neq y_1[n] + y_2[n]
\end{align*}
\]

...so adding a constant is a **nonlinear system**.
What about the real world?

Suppose you’re showing people images $x[n]$, and measuring their brain activity $y[n]$ as a result. How can you tell if this system is linear?

- Show them one image, call it $x_1[n]$. Measure the resulting brain activity, $y_1[n]$.
- Show them another image, $x_2[n]$. Measure the brain activity, $y_2[n]$.
- Show them $x[n] = x_1[n] + x_2[n]$. Measure $y[n]$. Is it equal to $y_1[n] + y_2[n]$?
- Repeat this experiment with lots of different images, and their sums, until you are convinced that the system is linear (or not).
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A system $\mathcal{H}$ is **shift-invariant** if these two algorithms compute the same thing (here $\mathcal{T}$ means “time shift”):

$$
\begin{align*}
    x[n] & \xrightarrow{\mathcal{T}} x[n-1] & \xrightarrow{\mathcal{H}} y[n] & \xrightarrow{\mathcal{T}} y[n-1] \\
    x[n] & \xrightarrow{\mathcal{H}} y[n] & \xrightarrow{\mathcal{T}} ? 
\end{align*}
$$
A system $\mathcal{H}$ is said to be **shift-invariant** if and only if, for every $x_1[n]$,  

$$x_1[n] \xrightarrow{\mathcal{H}} y_1[n]$$

implies that  

$$x[n] = x_1[n - n_0] \xrightarrow{\mathcal{H}} y[n] = y_1[n - n_0]$$

In words: a system is **shift-invariant** if and only if, for any input $x_1[n]$, (1) shifting the input by some number of samples $n_0$, and then passing it through the system, gives exactly the same result as (2) passing it through the system, and then shifting it.
Example: Averager

Let’s try it with the weighted averager.

\[
x_1[n] \xrightarrow{A} y_1[n] = \sum_{m=0}^{6} g[m]x_1[n - m]
\]

Then:

\[
x[n] = x_1[n - n_0] \xrightarrow{A} y[n] = \sum_{m=0}^{6} g[m]x[n - m]
\]

\[
= \sum_{m=0}^{6} g[m]x_1[(n - m) - n_0]
\]

\[
= \sum_{m=0}^{6} g[m]x_1[(n - n_0) - m]
\]

\[
= y_1[n - n_0]
\]

...so a weighted averager is a **shift-invariant system**.
Example: Square

Squaring the input is a nonlinear operation, but is it shift-invariant? Let’s find out:

\[ x_1[n] \xrightarrow{S} y_1[n] = x_1^2[n] \]

Then:

\[
\begin{align*}
x[n] &= x_1[n - n_0] \xrightarrow{A} y[n] = x^2[n] \\
&= (x_1[n - n_0])^2 \\
&= x_1^2[n - n_0] \\
&= y_1[n - n_0]
\end{align*}
\]

...so computing the square is a **shift-invariant system**.
Example: Windowing

How about windowing, e.g., in order to create an image?

\[ x_1[n] \xrightarrow{\mathcal{W}} y_1[n] = \begin{cases} x_1[n] & 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases} \]

If we shift the output, we get

\[ y_1[n - n_0] = \begin{cases} x_1[n - n_0] & n_0 \leq n \leq N - 1 + n_0 \\ 0 & \text{otherwise} \end{cases} \]

...but if we shift the input \( x[n] = x_1[n - n_0] \), we get

\[ y[n] = \begin{cases} x[n] & 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} x_1[n - n_0] & 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ \neq y_1[n - n_0] \]

...so windowing is a **shift-varying** system (not shift-invariant).
How about the real world?

Suppose you’re showing images \( x[n] \), and measuring the neural response \( y[n] \). How do you determine if this system is shift-invariant?

- Show an image \( x_1[n] \), and measure the neural response \( y_1[n] \).
- Shift the image by \( n_0 \) columns to the right, to get the image \( x[n] = x_1[n - n_0] \). Show people \( x[n] \).
- Is the resulting neural response exactly the same, but shifted to a slightly different set of neurons (shifted “to the right?”) If so, then the system may be shift-invariant!
- Keep doing that, with many different images and many different shifts, until you’re convinced the system is shift-invariant.
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Prove that differentiation, \( y(t) = \frac{dx}{dt} \), is a linear shift-invariant system (in terms of \( t \) as the time index, instead of \( n \)).
A system is **linear** if and only if, for any two inputs \( x_1[n] \) and \( x_2[n] \) that produce outputs \( y_1[n] \) and \( y_2[n] \),

\[
x[n] = x_1[n] + x_2[n] \xrightarrow{H} y[n] = y_1[n] + y_2[n]
\]

A system is **shift-invariant** if and only if, for any input \( x_1[n] \) that produces output \( y_1[n] \),

\[
x[n] = x_1[n - n_0] \xrightarrow{H} y[n] = y_1[n - n_0]
\]