Lecture 9: Exam 1 Review

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ECE 401: Signal and Image Analysis, Fall 2022
1. Topics Covered
2. Phasors
3. Spectrum
4. Fourier Series
5. Sampling and Interpolation
6. Summary
Outline

1. Topics Covered

2. Phasors

3. Spectrum

4. Fourier Series

5. Sampling and Interpolation

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### Topics Covered

1. HW1: Phasors
2. MP1: Spectrum
3. HW2: Fourier Series
4. MP2: Sampling
Outline

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Phasors

\[ x(t) = A \cos(2\pi ft + \theta) \]

\[ = \Re \{ z e^{j2\pi ft} \} \]

\[ = \frac{1}{2} z^* e^{-j2\pi ft} + \frac{1}{2} z e^{j2\pi ft} \]

where

\[ z = Ae^{j\theta} \]
Adding Phasors

How do you add

\[ z(t) = A \cos(2\pi ft + \theta) + B \cos(2\pi ft + \phi) \]?

Answer:

\[ z = (A \cos \theta + B \cos \phi) + j(A \sin \theta + B \sin \phi) \]

\[ z(t) = \Re \left\{ ze^{j2\pi ft} \right\} \]
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Two-sided spectrum

The spectrum of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}(x(t)) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$
The **spectrum plot** of a periodic signal is a plot with

- frequency on the X-axis,
- showing a vertical spike at each frequency component,
- each of which is labeled with the corresponding phasor.
Example: Cosine w/Amplitude 3, Phase $\pi/4$

$$x(t) = 3\cos(2\pi 800t + \pi/4)$$

Spectrum of $x(t)$

$$\frac{3}{2}e^{-jn/4}$$

$$\frac{3}{2}e^{jn/4}$$
Property #1: Scaling

Suppose we have a signal

\[ x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t} \]

Suppose we scale it by a factor of \( G \):

\[ y(t) = Gx(t) \]

That just means that we scale each of the coefficients by \( G \):

\[ y(t) = \sum_{k=-N}^{N} (Ga_k) e^{j2\pi f_k t} \]
Property #2: Adding a constant

Suppose we have a signal

\[ x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t} \]

Suppose we add a constant, \( C \):

\[ y(t) = x(t) + C \]

That just means that we add that constant to \( a_0 \):

\[ y(t) = (a_0 + C) + \sum_{k \neq 0} a_k e^{j2\pi f_k t} \]
Property #3: Adding two signals

Suppose we have two signals:

\[ x(t) = \sum_{n=-N}^{N} a'_n e^{j2\pi f'_n t} \]

\[ y(t) = \sum_{m=-M}^{M} a''_m e^{j2\pi f''_m t} \]

and we add them together:

\[ z(t) = x(t) + y(t) = \sum_{k} a_k e^{j2\pi f_k t} \]

where, if a frequency \( f_k \) comes from both \( x(t) \) and \( y(t) \), then we have to do phasor addition:

If \( f_k = f'_n = f''_m \) then \( a_k = a'_n + a''_m \)
Property #4: Time shift

Suppose we have a signal

\[ x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t} \]

and we want to time shift it by \( \tau \) seconds:

\[ y(t) = x(t - \tau) \]

Time shift corresponds to a phase shift of each spectral component:

\[ y(t) = \sum_{k=-N}^{N} \left( a_k e^{-j2\pi f_k \tau} \right) e^{j2\pi f_k t} \]
Property #5: Frequency shift

Suppose we have a signal

\[ x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t} \]

and we want to shift it in frequency by some constant overall shift, \( F \):

\[ y(t) = \sum_{k=-N}^{N} a_k e^{j2\pi (f_k + F) t} \]

Frequency shift corresponds to amplitude modulation (multiplying it by a complex exponential at the carrier frequency \( F \)):

\[ y(t) = x(t) e^{j2\pi F t} \]
Property #6: Differentiation

Suppose we have a signal

\[ x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t} \]

and we want to differentiate it:

\[ y(t) = \frac{dx}{dt} \]

Differentiation corresponds to scaling each spectral coefficient by \( j2\pi f_k \):

\[ y(t) = \sum_{k=-N}^{N} (j2\pi f_k a_k) e^{j2\pi f_k t} \]
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Fourier Series

- **Analysis** (finding the spectrum, given the signal):
  \[ X_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0} \, dt \]

- **Synthesis** (finding the signal, given the spectrum):
  \[ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} \]
Discrete-Time Fourier Series

- **Analysis** (finding the spectrum, given the signal):
  \[
  X_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j2\pi kn/N_0}
  \]

- **Synthesis** (finding the signal, given the spectrum):
  \[
  x[n] = \sum_k X_k e^{j2\pi kn/N_0}
  \]
  where the sum is over any set of \(N_0\) consecutive harmonics.
Spectral Properties of Fourier Series

- **Scaling:**
  \[ y(t) = Gx(t) \Leftrightarrow Y_k = GX_k \]

- **Add a Constant:**
  \[ y(t) = x(t) + C \Leftrightarrow Y_k = \begin{cases} X_0 + C & k = 0 \\ X_k & \text{otherwise} \end{cases} \]

- **Add Signals:** Suppose that \( x(t) \) and \( y(t) \) have the same fundamental frequency, then
  \[ z(t) = x(t) + y(t) \Leftrightarrow Z_k = X_k + Y_k \]
Spectral Properties of Fourier Series

• **Time Shift:** Shifting to the right, in time, by $\tau$ seconds:
  \[
y(t) = x(t - \tau) \iff Y_k = a_k e^{-j2\pi kF_0 \tau}
  \]

• **Frequency Shift:** Shifting upward in frequency by $F$ Hertz:
  \[
y(t) = x(t)e^{j2\pi dF_0 t} \iff Y_k = X_{k-d}
  \]

• **Differentiation:**
  \[
y(t) = \frac{dx}{dt} \iff Y_k = j2\pi kF_0 X_k
  \]
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How to sample a continuous-time signal

Suppose you have some continuous-time signal, $x(t)$, and you’d like to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{F_s}$ seconds:

$$x[n] = x(t = nT_s)$$
The spectrum plot of a **discrete-time periodic signal** is a regular spectrum plot, but with the X-axis relabeled. Instead of frequency in Hertz = \( \frac{\text{cycles}}{\text{second}} \), we use

\[
\omega \left[ \frac{\text{radians}}{\text{sample}} \right] = \frac{2\pi \left[ \frac{\text{radians}}{\text{cycle}} \right] f \left[ \frac{\text{cycles}}{\text{second}} \right]}{F_s \left[ \frac{\text{samples}}{\text{second}} \right]}
\]
Example: Cosine w/Amplitude 3, Phase $\pi/4$

$$x(t) = 3\cos(\pi/4 + 2\pi800n/8000) = 3\cos(\pi/4 + \pi n/5)$$
A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, \( f < \frac{F_s}{2} \).

If the Nyquist criterion is violated, then:

- If \( \frac{F_s}{2} < f < F_s \), then it will be aliased to
  \[
  f_a = F_s - f \\
  z_a = z^*
  \]
  i.e., the sign of all sines will be reversed.

- If \( F_s < f < \frac{3F_s}{2} \), then it will be aliased to
  \[
  f_a = f - F_s \\
  z_a = z
  \]
Example: Cosine with Amplitude 3, Phase $\pi/4$

$$x(t) = 3\cos(\pi/4 + 2\pi 4800n/8000) = 3\cos(\pi/4 + 6\pi n/5) = 3\cos(-\pi/4 + 4\pi n/5)$$

**Spectrum of $x[n]$**

- $(3/2)e^{-jn/4}$
- $(3/2)e^{jn/4}$
- $(3/2)e^{-jn/4}$
- $(3/2)e^{jn/4}$
Interpolation is the general method for reconstructing a continuous-time signal from its samples. The formula is:

\[ y(t) = \sum_{n=-\infty}^{\infty} y[n] p(t - nT_s) \]
Interpolation kernels

- Piece-wise constant interpolation = interpolate using a rectangle
- Piece-wise linear interpolation = interpolate using a triangle
- Ideal interpolation = interpolate using a sinc
Rectangular pulses

For example, suppose that the pulse is just a rectangle,

\[ p(t) = \begin{cases} 
1 & -\frac{T_s}{2} \leq t < \frac{T_s}{2} \\
0 & \text{otherwise}
\end{cases} \]
The result is a piece-wise constant interpolation of the digital signal:
The rectangular pulse has the disadvantage that $y(t)$ is discontinuous. We can eliminate the discontinuities by using a triangular pulse:

$$p(t) = \begin{cases} 
1 - \frac{|t|}{T_s} & -T_s \leq t < T_s \\
0 & \text{otherwise}
\end{cases}$$
Triangular pulses = Piece-wise linear interpolation

The result is a piece-wise linear interpolation of the digital signal:
Sinc pulses

If a signal has all its energy at frequencies below Nyquist \( f < \frac{F_s}{2} \), then it can be perfectly reconstructed using sinc interpolation:

\[
p(t) = \frac{\sin(\pi t / T_S)}{\pi t / T_S}
\]
Sinc pulse = ideal bandlimited interpolation

If a signal has all its energy at frequencies below Nyquist \((f < \frac{F_s}{2})\), then it can be perfectly reconstructed using sinc interpolation:
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