Lecture 8: Sampling Theorem

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ECE 401: Signal and Image Analysis, Fall 2022
1. Review: Sampling

2. Spectrum Plots

3. Spectrum of Oversampled Signals

4. Spectrum of Undersampled Signals

5. The Sampling Theorem

6. Summary

7. Written Example
Outline

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How to sample a continuous-time signal

Suppose you have some continuous-time signal, $x(t)$, and you’d like to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{F_s}$ seconds:

$$x[n] = x(t = nT_s)$$
Aliasing

- A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, \( f < \frac{F_s}{2} \).
- If the Nyquist criterion is violated, then:
  - If \( \frac{F_s}{2} < f < F_s \), then it will be aliased to
    \[
    f_a = F_s - f \\
    z_a = z^*
    \]
    i.e., the sign of all sines will be reversed.
  - If \( F_s < f < \frac{3F_s}{2} \), then it will be aliased to
    \[
    f_a = f - F_s \\
    z_a = z
    \]
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The **spectrum plot** of a periodic signal is a plot with
- frequency on the X-axis,
- showing a vertical spike at each frequency component,
- each of which is labeled with the corresponding phasor.
Example: Sine Wave

\[ x(t) = \sin(2\pi 800t) \]

\[ = \frac{1}{2j} e^{j2\pi 800t} - \frac{1}{2j} e^{-j2\pi 800t} \]

The spectrum of \( x(t) \) is \( \{(-800, -\frac{1}{2j}), (800, \frac{1}{2j})\} \).
Example: Sine Wave

$x(t) = \sin(2\pi 800t)$

Spectrum of $x(t)$

$(-1/2j)$  $(1/2j)$
Example: Quadrature Cosine

\[ x(t) = 3 \cos \left( 2\pi 800t + \frac{\pi}{4} \right) \]
\[ = \frac{3}{2} e^{j\pi/4} e^{j2\pi 800t} + \frac{3}{2} e^{-j\pi/4} e^{-j2\pi 800t} \]

The spectrum of \( x(t) \) is \( \{(-800, \frac{3}{2} e^{-j\pi/4}), (800, \frac{3}{2} e^{j\pi/4})\} \).
Example: Quadrature Cosine

\[ x(t) = 3\cos(2\pi 800t + \pi/4) \]

Spectrum of \( x(t) \):

- \( (3/2)e^{-j\pi/4} \)
- \( (3/2)e^{j\pi/4} \)
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A signal is called **oversampled** if $F_s > 2f$ (e.g., so that sinc interpolation can reconstruct it from its samples).
The spectrum plot of a **discrete-time periodic signal** is a regular spectrum plot, but with the X-axis relabeled. Instead of frequency in Hertz=$\frac{\text{cycles}}{\text{second}}$, we use

$$\omega \left[ \frac{\text{radians}}{\text{sample}} \right] = \frac{2\pi \left[ \frac{\text{radians}}{\text{cycle}} \right]}{F_s \left[ \frac{\text{samples}}{\text{second}} \right]} f \left[ \frac{\text{cycles}}{\text{second}} \right]$$
How do we plot the aliasing?

Remember that a discrete-time signal has energy at
- $f$ and $-f$, but also $F_s - f$ and $-F_s + f$, and $F_s + f$ and $-F_s - f$, and...
- $\omega$ and $-\omega$, but also $2\pi - \omega$ and $-2\pi + \omega$, and $2\pi + \omega$ and $-2\pi - \omega$, and...

Which ones should we plot? Answer: **plot all of them!** Usually we plot a few nearest the center, then add “…” at either end, to show that the plot continues forever.
Let’s sample at $F_s = 8000$ samples/second.

$$x[n] = \sin \left( 2\pi \frac{800n}{8000} \right)$$

$$= \sin \left( \frac{\pi n}{5} \right)$$

$$= \frac{1}{2j} e^{j\pi n/5} - \frac{1}{2j} e^{-j\pi n/5}$$

The spectrum of $x[n]$ is \{ ... \(-\pi/5, -\frac{1}{2j}\), \((\pi/5, \frac{1}{2j})\), ... \}.  

Example: Sine Wave

\[ x[n] = \sin\left(\frac{2\pi 800n}{8000}\right) = \sin\left(\frac{\pi n}{5}\right) \]
Example: Quadrature Cosine

\[ x[n] = 3 \cos \left( 2\pi \frac{800n}{8000} + \frac{\pi}{4} \right) \]
\[ = 3 \cos \left( \pi \frac{n}{5} + \frac{\pi}{4} \right) \]
\[ = \frac{3}{2} e^{j\pi/4} e^{j\pi n/5} + \frac{3}{2} e^{-j\pi/4} e^{-j\pi n/5} \]

The spectrum of \( x[n] \) is \{ \ldots, (-\pi/5, \frac{3}{2} e^{-j\pi/4}), (\pi/5, \frac{3}{2} e^{j\pi/4}), \ldots \}. \]
Example: Quadrature Cosine

\[ x(t) = 3\cos(\pi/4 + 2\pi 800n/8000) = 3\cos(\pi/4 + \pi n/5) \]
A signal is called **undersampled** if $F_s < 2f$ (e.g., so that sinc interpolation can’t reconstruct it from its samples).
... but Aliasing?

Remember that a discrete-time signal has energy at

- $f$ and $-f$, but also $F_s - f$ and $-F_s + f$, and $F_s + f$ and $-F_s - f$, and...
- $\omega$ and $-\omega$, but also $2\pi - \omega$ and $-2\pi + \omega$, and $2\pi + \omega$ and $-2\pi - \omega$, and...

We still want to plot all of these, but now $\omega$ and $-\omega$ won’t be the spikes closest to the center. Instead, some other spike will be closest to the center.
Example: Sine Wave

Let’s still sample at $F_s = 8000$, but we’ll use a sine wave at $f = 4800\text{Hz}$, so it gets undersampled.

$$x[n] = \sin \left( 2\pi \frac{4800n}{8000} \right)$$
$$= \sin \left( \frac{6\pi n}{5} \right)$$
$$= -\sin \left( \frac{4\pi n}{5} \right)$$
$$= -\frac{1}{2j} e^{j\frac{4\pi n}{5}} + \frac{1}{2j} e^{-j\frac{4\pi n}{5}}$$

The spectrum of $x[n]$ is $\{\ldots, (-4\pi/5, \frac{1}{2j}), (4\pi/5, -\frac{1}{2j}), \ldots\}$. 
Example: Sine Wave

\[ x[n] = \sin\left(\frac{2\pi 4800n}{8000}\right) = \sin\left(\frac{6\pi n}{5}\right) = -\sin\left(\frac{4\pi n}{5}\right) \]
Example: Quadrature Cosine

\[ x[n] = 3 \cos \left( \frac{2\pi 4800 n}{8000} + \frac{\pi}{4} \right) \]
\[ = 3 \cos \left( \frac{6\pi n}{5} + \frac{\pi}{4} \right) \]
\[ = 3 \cos \left( \frac{4\pi n}{5} - \frac{\pi}{4} \right) \]
\[ = \frac{3}{2} e^{-j\pi/4} e^{j4\pi n/5} + \frac{3}{2} e^{j\pi/4} e^{-j4\pi n/5} \]

The spectrum of \( x[n] \) is
\{\ldots, \left( -\frac{4\pi}{5}, \frac{3}{2} e^{j\pi/4} \right), \left( \frac{4\pi}{5}, \frac{3}{2} e^{-j\pi/4} \right), \ldots \}. \]
Example: Quadrature Cosine

\[ x(t) = 3\cos(\pi/4 + 2\pi 4800n/8000) = 3\cos(\pi/4 + 6\pi n/5) = 3\cos(-\pi/4 + 4\pi n/5) \]
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Let’s assume that $x(t)$ is periodic with some period $T_0$, therefore it has a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} = \sum_{k=0}^{\infty} |X_k| \cos \left( \frac{2\pi kt}{T_0} + \angle X_k \right)$$
Eliminate the aliased tones

We already know that $e^{j2\pi kt/T_0}$ will be aliased if $|k|/T_0 > F_N$. So let’s assume that the signal is **band-limited**: it contains no frequency components with frequencies larger than $F_S/2$. That means that the only $X_k$ with nonzero energy are the ones in the range $-N/2 < k < N/2$, where $N \leq F_S T_0$.

$$x(t) = \sum_{k=-N/2}^{N/2} X_k e^{j2\pi kt/T_0} = \sum_{k=0}^{N/2} |X_k| \cos \left( \frac{2\pi kt}{T_0} + \angle X_k \right)$$
Sample that signal!

Now let's sample that signal, at sampling frequency $F_S$:

$$x[n] = \sum_{k=-N/2}^{N/2} X_k e^{j2\pi kn/F_S T_0} = \sum_{k=0}^{N/2} |X_k| \cos \left( \frac{2\pi kn}{N} + \angle X_k \right)$$

So the highest digital frequency, when $k = F_S T_0/2$, is $\omega_k = \pi$. The lowest is $\omega_0 = 0$.

$$x[n] = \sum_{\omega_k=-\pi}^{\pi} X_k e^{j\omega_k n} = \sum_{\omega_k=0}^{\pi} |X_k| \cos (\omega_k n + \angle X_k)$$
Spectrum of a sampled periodic signal

\[ x(t) \text{ with frequencies up to 63kHz} \]

\[ x(t) \text{ with frequencies up to 7kHz} \]

\[ x[n] = x(n/16000) \]

\[ \text{Spectrum of } x(t) \text{ with frequencies up to } +/- 64 \text{ kHz} \]

\[ \text{Spectrum of } x(t) \text{ with frequencies up to } +/- 8 \text{ kHz} \]

\[ \text{Spectrum of } x[n] \]

\[ \text{Time (ms)} \quad 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \]

\[ \text{Frequency (kHz)} \quad -60 \quad -40 \quad -20 \quad 0 \quad 20 \quad 40 \quad 60 \]

\[ \text{Time (samples)} \quad 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \]

\[ \text{Frequency (radians/sample)} \quad -\pi \quad -3\pi/4 \quad -\pi/2 \quad -\pi/4 \quad \pi/4 \quad \pi/2 \quad 3\pi/4 \quad \pi \]
The sampling theorem

As long as $-\pi \leq \omega_k \leq \pi$, we can recreate the continuous-time signal by either (1) using sinc interpolation, or (2) regenerating a continuous-time signal with the corresponding frequency:

$$f_k \left[ \frac{\text{cycles}}{\text{second}} \right] = \frac{\omega_k \left[ \frac{\text{radians}}{\text{sample}} \right]}{2\pi \left[ \frac{\text{radians}}{\text{cycle}} \right]} \times F_s \left[ \frac{\text{samples}}{\text{second}} \right]$$

$$x[n] = \cos(\omega_k n + \theta_k) \rightarrow x(t) = \cos(2\pi f_k t + \theta_k)$$
The sampling theorem

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\text{max}}$ can be reconstructed exactly from its samples $x[n] = x(nT_S)$ if the samples are taken at a rate $F_s = 1/T_s$ that is $F_s \geq 2f_{\text{max}}$. 
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\[
\omega \left[ \frac{\text{radians}}{\text{sample}} \right] = \frac{2\pi \left[ \frac{\text{radians}}{\text{cycle}} \right]}{F_s \left[ \frac{\text{samples}}{\text{second}} \right]} \cdot f \left[ \frac{\text{cycles}}{\text{second}} \right]
\]
The sampling theorem

A continuous-time signal \( x(t) \) with frequencies no higher than \( f_{\text{max}} \) can be reconstructed exactly from its samples \( x[n] = x(nT_s) \) if the samples are taken at a rate \( F_s = 1/T_s \) that is \( F_s \geq 2f_{\text{max}} \).
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Let $x(t)$ be a sinusoid with some amplitude, some phase, and some frequency.

- Plot the spectrum of $x(t)$.
- Choose an $F_s$ that undersamples it. Plot the spectrum of $x[n]$.