Lecture 6: Sampling and Aliasing

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ECE 401: Signal and Image Analysis, Fall 2022
1. Review: Spectrum of continuous-time signals
2. Sampling
3. Aliasing
4. Aliased Frequency
5. Aliased Phase
6. Summary
7. Written Example
Outline

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Two-sided spectrum

The spectrum of \( x(t) \) is the set of frequencies, and their associated phasors,

\[
\text{Spectrum } (x(t)) = \{ (f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N) \}
\]
such that

\[
x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}
\]
One reason the spectrum is useful is that any periodic signal can be written as a sum of cosines. Fourier’s theorem says that any \( x(t) \) that is periodic, i.e.,

\[
x(t + T_0) = x(t)
\]

can be written as

\[
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kF_0 t}
\]

which is a special case of the spectrum for periodic signals: 
\( f_k = kF_0 \), and \( a_k = X_k \), and

\[
F_0 = \frac{1}{T_0}
\]
Fourier Series

- **Analysis** (finding the spectrum, given the waveform):

  \[ X_k = \frac{1}{T_0} \int_{0}^{T_0} x(t) e^{-j2\pi kt/T_0} \, dt \]

- **Synthesis** (finding the waveform, given the spectrum):

  \[ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} \]
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Suppose you have some continuous-time signal, $x(t)$, and you’d like to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{F_s}$ seconds:

$$x[n] = x(t = nT_s)$$
Example: a 1kHz sine wave

For example, suppose \( x(t) = \sin(2\pi 1000t) \). By sampling at \( F_s = 16000 \) samples/second, we get

\[
x[n] = \sin \left( 2\pi 1000 \frac{n}{16000} \right) = \sin(\pi n/8)
\]
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The question immediately arises: can every sine wave be reconstructed from its samples? The answer, unfortunately, is “no.”
Can every sine wave be reconstructed from its samples?

For example, two signals $x_1(t)$ and $x_2(t)$, at 10kHz and 6kHz respectively:

$$x_1(t) = \cos(2\pi 10000t), \quad x_2(t) = \cos(2\pi 6000t)$$

Let's sample them at $F_s = 16,000$ samples/second:

$$x_1[n] = \cos \left( 2\pi 10000 \frac{n}{16000} \right), \quad x_2[n] = \cos \left( 2\pi 6000 \frac{n}{16000} \right)$$

Simplifying a bit, we discover that $x_1[n] = x_2[n]$. We say that the 10kHz tone has been “aliased” to 6kHz:

$$x_1[n] = \cos \left( \frac{5\pi n}{4} \right) = \cos \left( \frac{3\pi n}{4} \right)$$

$$x_2[n] = \cos \left( \frac{3\pi n}{4} \right) = \cos \left( \frac{5\pi n}{4} \right)$$
Can every sine wave be reconstructed from its samples?
**What is the highest frequency that can be reconstructed?**

The highest frequency whose cosine can be exactly reconstructed from its samples is called the “Nyquist frequency,” \( F_N = F_S / 2 \). If \( x(t) = \cos(2\pi F_N t) \), then

\[
x[n] = \cos \left( 2\pi F_N \frac{n}{F_S} \right) = \cos(\pi n) = (-1)^n
\]
If you try to sample a signal whose frequency is above Nyquist (like the one shown on the left), then it gets aliased to a frequency below Nyquist (like the one shown on the right).

Continuous-time signal $x(t) = \cos(2\pi10000t)$

Discrete-time signal $x[n] = \cos(2\pi10000n/16000) = \cos(5\pi n/14) = \cos(3\pi n/4)$

Continuous-time signal $x(t) = \cos(2\pi6000t)$

Discrete-time signal $x[n] = \cos(2\pi6000n/16000) = \cos(3\pi n/4) = \cos(5\pi n/4)$
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Suppose you have a cosine at frequency $f$:

$$x(t) = \cos(2\pi ft)$$

Suppose you sample it at $F_s$ samples/second. If $F_s$ is not high enough, it might get aliased to some other frequency, $f_a$.

$$x[n] = \cos(2\pi fn/F_s) = \cos(2\pi f_a n/F_s)$$

How can you predict what $f_a$ will be?
Aliasing comes from two sources:

\[
\cos(\phi) = \cos(2\pi n - \phi)
\]
\[
\cos(\phi) = \cos(\phi - 2\pi n)
\]

The equations above are true for any integer \( n \).
Let’s plug in $\phi = \frac{2\pi fn}{F_s}$, and $2\pi = \frac{2\pi F_s}{F_s}$. That gives us:

$$\cos\left(\frac{2\pi fn}{F_s}\right) = \cos\left(\frac{2\pi n(F_s - f)}{F_s}\right)$$

$$\cos\left(\frac{2\pi fn}{F_s}\right) = \cos\left(\frac{2\pi (f - F_s)n}{F_s}\right)$$

So a discrete-time cosine at frequency $f$ is also a cosine at frequency $F_s - f$, and it’s also a cosine at $f - F_s$. 
A discrete-time cosine at frequency $f$ is also a cosine at frequency $F_s - f$, and it’s also a cosine at $f - F_s$.

So which of those frequencies will we hear when we play the sinusoid back again?

**Answer:** any frequency that can be reconstructed by the analog-to-digital converter. That means any frequency below the Nyquist frequency, $F_N = F_s/2$. 

Aliased Frequency

4Hz, at $f_s = 9$ Hz, looks like 4Hz

4Hz, at $f_s = 8$ Hz, looks like 4Hz

4Hz, at $f_s = 7$ Hz, looks like 3Hz

4Hz, at $f_s = 6$ Hz, looks like 2Hz

4Hz, at $f_s = 5$ Hz, looks like 1Hz
Aliased Frequency

4Hz, at $F_s=4.5$ Hz, looks like 0.5Hz

4Hz, at $F_s=4$ Hz, looks like 0Hz

4Hz, at $F_s=3.5$ Hz, looks like 0.5Hz

4Hz, at $F_s=3$ Hz, looks like 1Hz

4Hz, at $F_s=2.5$ Hz, looks like 1Hz
Aliased Frequency

All of the following frequencies are actually the same frequency when a cosine is sampled at \( F_s \) samples/second.

\[
f_a \in \{ f - \ell F_s, \ell F_s - f : \ell \in \text{any integer} \}
\]

The “aliased frequency” is whichever of those is below Nyquist \((F_s/2)\). Usually there’s only one that’s below Nyquist, so you can just look for

\[
f_a = \min (f - \ell F_s, \ell F_s - f : \ell \in \text{any integer})
\]
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Sine waves are different for the following reason:

\[
\sin(\phi) = -\sin(2\pi n - \phi) \\
\sin(\phi) = \sin(\phi - 2\pi n)
\]
Therefore:

\[
\sin \left( \frac{2\pi fn}{F_s} \right) = - \sin \left( \frac{2\pi n(F_s - f)}{F_s} \right)
\]

\[
\sin \left( \frac{2\pi fn}{F_s} \right) = \sin \left( \frac{2\pi (f - F_s)n}{F_s} \right)
\]

So a discrete-time sine at frequency \( f \) is also a **negative** sine at frequency \( F_s - f \), and a **positive** sine at frequency \( f - F_s \).
Sine is Different

4Hz sine, at $F_s=10$ Hz, looks like 4Hz

4Hz sine, at $F_s=9$ Hz, looks like 4Hz

4Hz sine, at $F_s=8$ Hz, looks like 4Hz

4Hz sine, at $F_s=7$ Hz, looks like 3Hz

4Hz sine, at $F_s=6$ Hz, looks like 2Hz
Aliased Phase of a General Phasor

For a general complex exponential, we get:

\[ ze^{j\phi} = ze^{j(\phi - 2\pi n)} = \left( z^* e^{j(2\pi n - \phi)} \right)^* \]

Therefore:

\[ \Re \left\{ ze^{j \frac{2\pi fn}{F_s}} \right\} = \Re \left\{ ze^{j \frac{2\pi (f - F_s)n}{F_s}} \right\} = \Re \left\{ z^* e^{j \frac{2\pi (F_s - f)n}{F_s}} \right\} \]
Aliased Phase of a General Phasor

Suppose we have some frequency $f$, and we’re trying to find its aliased frequency $f_a$.

- Among the several possibilities, if $f_a = F_s - f$ is below Nyquist, then that’s the frequency we’ll hear. Its phasor will be the complex conjugate of the original phasor,

  $$Z_a = Z^*$$

- On the other hand, if $f_a = f - F_s$ is below Nyquist, then that’s the frequency we’ll hear. Its phasor will be the same as the phasor of the original sinusoid:

  $$Z_a = Z$$
Aliased Phase of a General Phasor

4Hz at $-\pi/4$ phase, at $F_s=30$ Hz, looks like 4Hz with phase of $-\pi/4$

4Hz at $-\pi/4$ phase, at $F_s=11$ Hz, looks like 4Hz with phase of $-\pi/4$

4Hz at $-\pi/4$ phase, at $F_s=10$ Hz, looks like 4Hz with phase of $-\pi/4$

4Hz at $-\pi/4$ phase, at $F_s=6$ Hz, looks like 2Hz with phase of $+\pi/4$

4Hz at $-\pi/4$ phase, at $F_s=5$ Hz, looks like 1Hz with phase of $+\pi/4$
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A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, \( f < \frac{F_s}{2} \).

If the Nyquist criterion is violated, then:

- If \( \frac{F_s}{2} < f < F_s \), then it will be aliased to
  \[
  f_a = F_s - f \\
  z_a = z^*
  \]
  i.e., the sign of all sines will be reversed.
- If \( F_s < f < \frac{3F_s}{2} \), then it will be aliased to
  \[
  f_a = f - F_s \\
  z_a = z
  \]
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Sketch a sinusoid with some arbitrary phase (say, $-\pi/4$). Show where the samples are if it’s sampled:

- more than twice per period
- more than once per period, but less than twice per period
- less than once per period