Lecture 3: Spectrum

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ECE 401: Signal and Image Analysis, Fall 2022

Outline

1. Beat Tones
2. Spectrum
3. Periodic Signals
4. Properties of a Fourier Spectrum
5. Summary

Beat tones

When two pure tones at similar frequencies are added together, you hear the two tones "beating" against each other.

Beat tones demo

Beat tones and Trigonometric identities

Beat tones can be explained using this trigonometric identity:

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

Let's do the following variable substitution:

$$\begin{aligned} a+b &= 2\pi f_1 t \\ a-b &= 2\pi f_2 t \\ a &= 2\pi f_{ave} t \\ b &= 2\pi f_{beat} t \end{aligned}$$

where $f_{ave} = \frac{f_1+f_2}{2}$, and $f_{beat} = \frac{f_1-f_2}{2}$.

Beat tones and Trigonometric identities

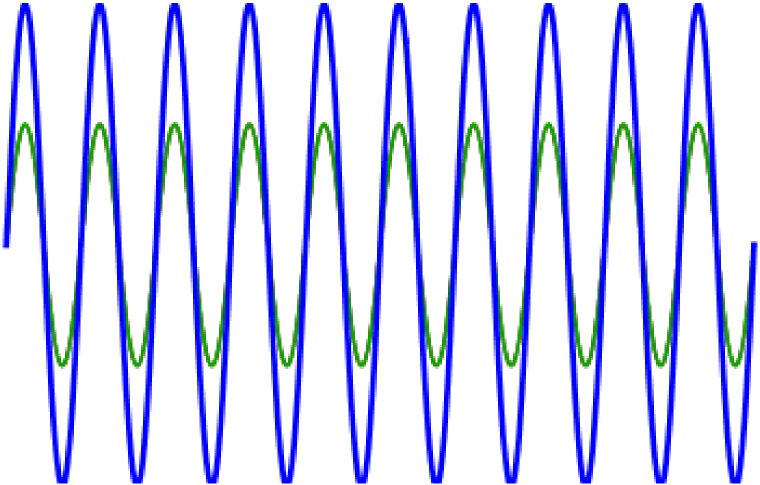
Re-writing the trigonometric identity, we get:

$$\frac{1}{2}\cos(2\pi f_1 t) + \frac{1}{2}\cos(2\pi f_2 t) = \cos(2\pi f_{beat} t)\cos(2\pi f_{ave} t)$$

So when we play two tones together, $f_1 = 110$Hz and $f_2 = 104$Hz, it sounds like we're playing a single tone at $f_{ave} = 107$Hz, multiplied by a beat frequency $f_{beat} = 3$ (double beats)/second.

Beat tones

by Adjwilley, CC-SA 3.0, https://commons.wikimedia.org/wiki/File:WaveInterference.gif



More complex beat tones

What happens if we add together, say, three tones?

$$\cos(2\pi 107t) + \cos(2\pi 110t) + \cos(2\pi 104t) = ???$$

For this, and other more complicated operations, it is much, much easier to work with complex exponentials, instead of cosines.

More complex beat tones

What happens if we add together, say, three tones?

$$x(t) = \cos(2\pi 107 t) + \cos(2\pi 110 t) + \cos(2\pi 104 t) = ???$$

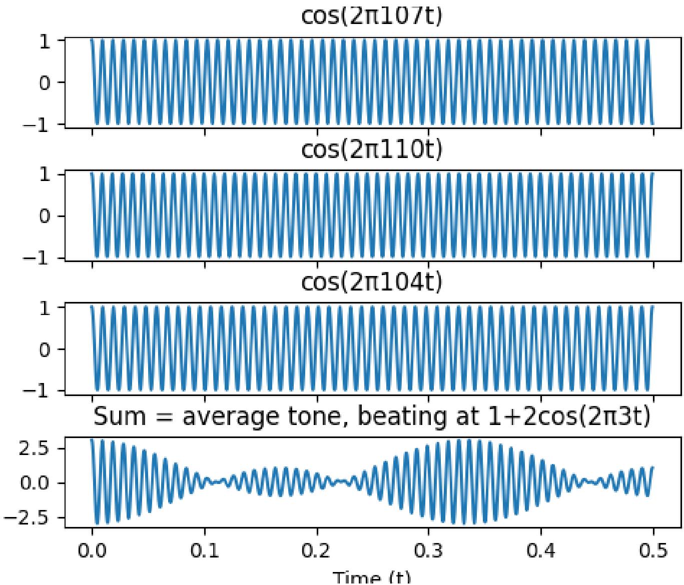
This is like a phasor example, except that all of the tones are at different frequencies.

$$\begin{aligned} x(t) &= \Re\left\{e^{j2\pi 107t} + e^{j2\pi 110t} + e^{j2\pi 104t}\right\} \\ &= \Re\left\{\left(1 + e^{j2\pi 3t} + e^{-j2\pi 3t}\right) e^{j2\pi 107t}\right\} \end{aligned}$$

So we just have to do this phasor addition:

$$1 + e^{j2\pi 3t} + e^{-j2\pi 3t} = 1 + 2\cos\left(2\pi 3t\right)$$

Triple-beat example



Outline

Phasor representation of a general sum of sinusoids

In general, if $x(t)$ is a sum of sines and cosines,

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos\left(2\pi f_k t + \theta_k\right)$$

Then it has a phasor notation

$$x(t) = A_0 + \sum_{k=1}^{N} \Re\left\{A_k e^{j\theta_k} e^{j2\pi f_k t}\right\}$$

Two-sided spectrum

The $\Re\{z\}$ operator is annoying. In order to get rid of it, let's take advantage of Euler's formula $\Re\{z\} = \frac{1}{2}(z + z^*)$ to write:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k)$$
$$= \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

In order to do that, we need to define a_k like this:

$$a_k = \begin{cases} A_0 & k = 0 \\ \frac{1}{2} A_k e^{j\theta_k} & k > 0 \\ \frac{1}{2} A_{-k} e^{-j\theta_{-k}} & k < 0 \end{cases}$$

Two-sided spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\left(x(t)\right) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Outline

Fourier's theorem

One reason the spectrum is useful is that **any** periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t+T_0) = x(t)$$

can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

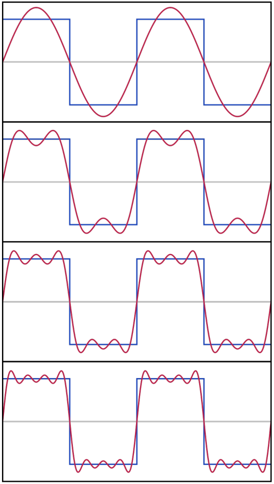
which is a special case of the spectrum for periodic signals: $f_k = kF_0$, and $a_k = X_k$, and

$$F_0 = \frac{1}{T_0}$$

Analysis and Synthesis

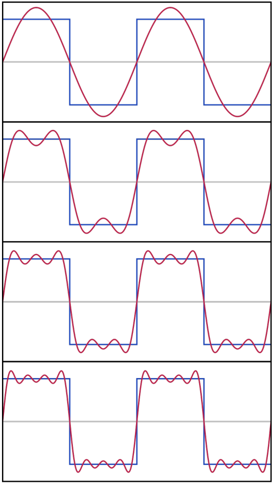
- **Fourier Analysis** is the process of finding the spectrum, X_k, given the signal $x(t)$. I'll tell you how to do that next lecture.
- **Fourier Synthesis** is the process of generating the signal, $x(t)$, given its spectrum. I'll spend the rest of today's lecture showing examples and properties of synthesis.

Example: Square wave



Jim.belk, Public domain image 2009, https://commons.wikimedia.org/wiki/File:Fourier_Series.svg

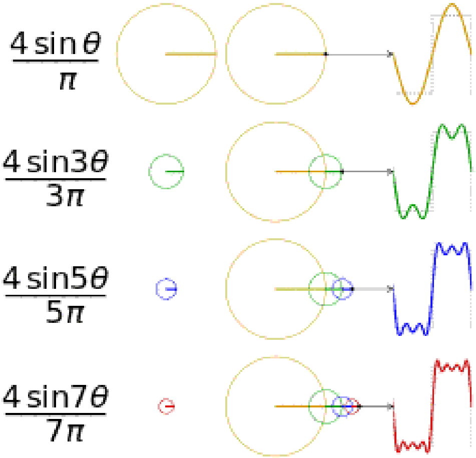
Example #1: Square wave



Jim.belk, Public domain image 2009, `https://commons.wikimedia.org/wiki/File:Fourier_Series.svg`

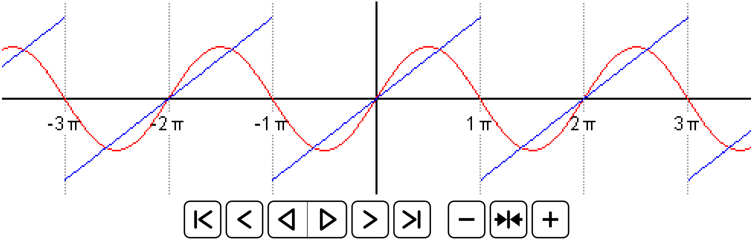
Example #1: Square wave

https://upload.wikimedia.org/wikipedia/commons/b/bd/Fourier_series_square_wave_circles_animation.svg



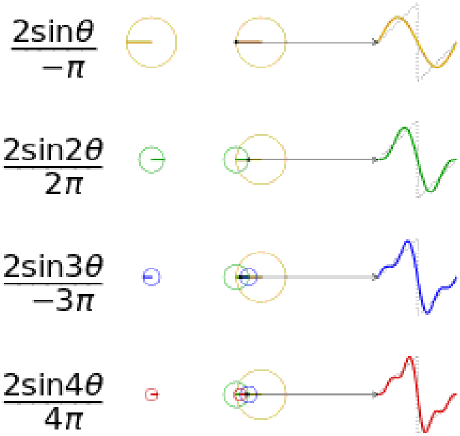
Example #2: Sawtooth wave

By Lucas Vieira, public domain 2009, https://commons.wikimedia.org/wiki/File:Periodic_identity_function.gif



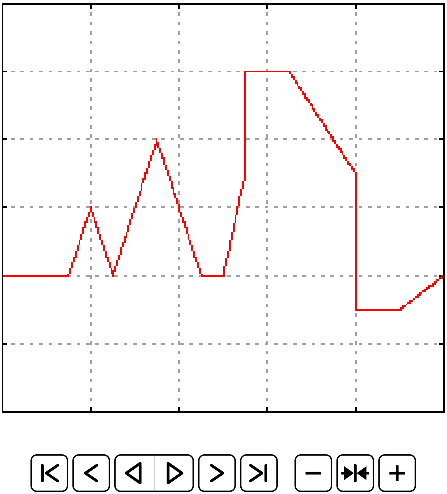
Example #2: Sawtooth wave

https://upload.wikimedia.org/wikipedia/commons/1/1e/Fourier_series_sawtooth_wave_circles_animation.svg

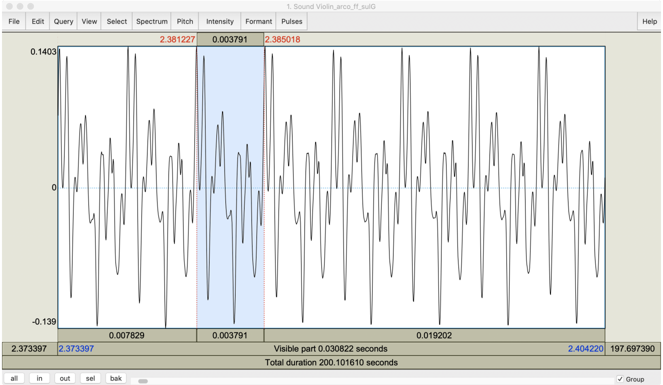


Example: A weird arbitrary signal

By Scallop7, CC-SA 4.0 2007, https://commons.wikimedia.org/wiki/File:Example_of_Fourier_Convergence.gif

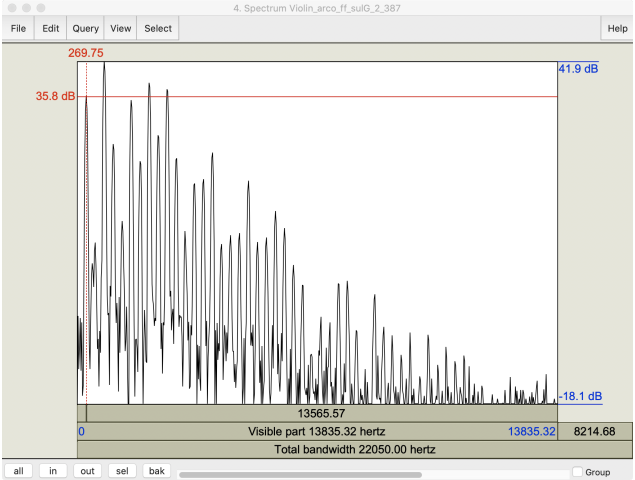


Example: Violin



Eight periods from the recording of a violin playing $f = 1/0.003791 = 262$Hz, i.e., C4 (middle C). Waveform distributed by **University of Iowa Electronic Music Studios**.

Example: Violin



Log magnitude spectrum ($20 \log_{10} |X_k|$) for the first 43 harmonics or so ($1 \leq k \leq 43$ or so) of a violin playing C4. Waveform distributed by **University of Iowa Electronic Music Studios**.

Outline

Properties of a spectrum

Spectrum representation is nice to use because

- It's so general. Any periodic signal can be written this way.
- Many signal processing operations can be written directly in the spectral domain (as operations on a_k), without converting back to $x(t)$.

Property #1: Scaling

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Suppose we scale it by a factor of G:

$$y(t) = Gx(t)$$

That just means that we scale each of the coefficients by G:

$$y(t) = \sum_{k=-N}^{N} (Ga_k)\, e^{j2\pi f_k t}$$

Property #2: Adding a constant

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Suppose we add a constant, C:

$$y(t) = x(t) + C$$

That just means that we add that constant to a_0:

$$y(t) = (a_0 + C) + \sum_{k \neq 0} a_k e^{j2\pi f_k t}$$

Property #3: Adding two signals

Suppose we have two signals:

$$x(t) = \sum_{n=-N}^{N} a'_n e^{j2\pi f'_n t}$$

$$y(t) = \sum_{m=-M}^{M} a''_m e^{j2\pi f''_m t}$$

and we add them together:

$$z(t) = x(t) + y(t) = \sum_k a_k e^{j2\pi f_k t}$$

where, if a frequency f_k comes from both $x(t)$ and $y(t)$, then we have to do phasor addition:

$$\text{If } f_k = f'_n = f''_m \text{ then } a_k = a'_n + a''_m$$

Property #4: Time shift

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

and we want to time shift it by τ seconds:

$$y(t) = x(t-\tau)$$

Time shift corresponds to a **phase shift** of each spectral component:

$$y(t) = \sum_{k=-N}^{N} \left(a_k e^{-j2\pi f_k \tau}\right) e^{j2\pi f_k t}$$

Property #5: Frequency shift

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

and we want to shift it in frequency by some constant overall shift, F:

$$y(t) = \sum_{k=-N}^{N} a_k e^{j2\pi (f_k+F)t}$$

Frequency shift corresponds to amplitude modulation (multiplying it by a complex exponential at the carrier frequency F):

$$y(t) = x(t)e^{j2\pi Ft}$$

Property #6: Differentiation

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

and we want to differentiate it:

$$y(t) \propto \frac{dx}{dt}$$

Differentiation corresponds to scaling each spectral coefficient by $j2\pi f_k$:

$$y(t) = \sum_{k=-N}^{N} \left(j2\pi f_k a_k\right) e^{j2\pi f_k t}$$

Outline

Summary

- **Spectrum:** The spectrum of any sum of cosines is the set of complex-valued spectral coefficients, a_k, matched with the frequencies f_k, such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

- **Fourier's Theorem:** Any periodic waveform, $x(t+T_0) = x(t)$, can be synthesized as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

- **Properties of the spectrum:** signal processing operations that can be done directly in the spectrum, without first recomputing the waveform, include scaling, adding, time shift, frequency shift, and differentiation.