ECE 401 Signal and Image Analysis
Homework 3

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Assigned: 9/28/2022; Due: 10/5/2022
Reading: DSP First Chapter 5

Problem 3.1

In MP3, one of the filters you’ll create is a local averaging filter. A local averaging filter produces an output $y[n]$, at time $n$, which is the average of the previous $N$ samples of $x[m]$:  
\[ y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \quad (3.1-1) \]

\[ h[m] = \begin{cases} 1/N & 0 \leq m \leq N-1 \\ 0 & \text{otherwise} \end{cases} \quad (3.1-2) \]

(a) First, consider what happens if $x[m]$ is a pure tone with a period of $N_0 = \frac{2\pi}{\omega_0}$, an amplitude of $A$, and a phase of $\theta$:

\[ x[n] = A \cos(\omega_0 n - \theta) \]

Suppose that the averaging window, $N$, is exactly an integer multiple of $N_0$. For example, suppose that $N = 3N_0$. Draw a picture of $x[n]$ as a function of $n$, and shade in the regions that would be added together by the summation in Eq. (3.1-1) in order to compute $y[0]$. Argue based on your figure (with no calculations at all) that $y[0] = 0$.

Solution: The picture should show that we are adding together three complete periods of the cosine up through, and including, the sample $x[0]$. Every period of the cosine has a positive section and a negative section. When we average these two sections, they cancel each other out.

(b) Adding up the samples of a cosine is easy when $N$ is an integer multiple of $N_0$, but hard otherwise. It’s actually much easier to add the samples of a complex exponential, because we can use the standard geometric series formula (https://en.wikipedia.org/wiki/Geometric_series#Formula). Use that formula to find $y[N-1]$ when \[ x[n] = e^{j\omega_0 n} \]

Your result should have the form $y[0] = (1 - a)/(1 - b)$ for some complex-valued constants, $a$ and $b$, that depend on $\pi$, $N$, and $\omega_0$, but not on $m$ or $n$.

Solution:

\[ y[0] = \frac{1}{N} \left( \frac{1 - e^{-j\omega_0 N}}{1 - e^{-j\omega_0}} \right) \]
Problem 3.2

Another of the filters you’ll create in MP3 is a backward-difference filter. A backward-difference filter is one of several different ways of approximating a first-derivative:

\[ y[n] = \sum_{m=-\infty}^{\infty} h[m] x[n - m] \]  

(3.2-1)

\[ h[m] = \begin{cases} 
1 & m = 0 \\
-1 & m = 1 \\
0 & \text{otherwise}
\end{cases} \]  

(3.2-2)

(a) First, consider what happens if \( x[m] \) is a pure tone with a period of \( T_0 = \frac{2\pi}{\omega_0} \), an amplitude of \( A \), and a phase of \( \theta \):

\[ x[n] = A \cos (\omega_0 n - \theta) \]  

(3.2-3)

Plug Eq. (3.2-3) into Eq. (3.2-1), then use the following trigonometric identity and the following approximation to discover in exactly what way the first-difference operator approximates a derivative:

\[ -2 \sin(a) \sin(b) = \cos(a + b) - \cos(a - b) \]
\[ \sin(b) \approx b \text{ if } b \text{ is small} \]

In order to apply the approximation, you can assume that \( \omega_0 \) is a small number.

Solution: Plugging Eq. (3.2-3) into Eq. (3.2-1) gives

\[ y[n] = x[n] - x[n - 1] \]
\[ = A \cos (\omega_0 n - \theta) - A \cos (\omega_0 (n - 1) - \theta) \]

We can apply the trig identity if we set

\[ a = \omega_0 (n - 0.5) - \theta \]
\[ b = 0.5 \omega_0 \]

which gives

\[ y[n] = -2A \sin(0.5\omega_0) \sin(\omega_0(n - 0.5) + \theta) \]

Applying the approximation gives

\[ y[n] \approx -A \omega_0 \sin(\omega_0(n - 0.5) + \theta), \]

which is what you would get if you differentiated \( A \cos(\omega_0(n - 0.5) + \theta) \) with respect to \( n \).

(b) Let’s try the same thing with a complex exponential. Plug the following value of \( x[n] \) into Eq. (3.2-1)

\[ x[n] = e^{j\omega_0 n}, \]

then assume that \( \omega_0 \) is a small number, and simplify using the approximation

\[ e^\phi \approx 1 + \phi \text{ if } \phi \text{ is small} \]

in order to get something that looks like \( dx[n]/dn \).

Solution:

\[ y[n] = x[n] - n[n - 1] \]
\[ = e^{j\omega_0 n} - e^{j\omega_0(n - 1)} \]
\[ = e^{j\omega_0 n}(1 - e^{-j\omega_0}) \]
\[ \approx j\omega_0 e^{-j\omega_0 n} \]

which is \( dx[n]/dn \).