Problem 2.1

Suppose that

\[ v(t) = 2 \cos(2\pi 880t) + 2 \sin(2\pi 1320t) \]

What is the fundamental frequency? What are the Fourier series coefficients, \( V_k \)?

**Solution:** The fundamental frequency is the GCD of 880 and 1320, which is \( F_0 = 440 \)Hz. The component at \( k = 2 \) \( (f = 2F_0 = 880) \) is

\[ 2 \cos(2\pi 2F_0 t) = e^{j2\pi 2F_0 t} + e^{-j2\pi 2F_0 t} \]

so \( V_2 = V_{-2} = 1 \). The component at \( k = 3 \) \( (f = 3F_0 = 1320) \) is

\[ 2 \sin(2\pi 3F_0 t) = 2 \left( \frac{e^{j2\pi 3F_0 t} - e^{-j2\pi 3F_0 t}}{2j} \right) \]

so \( V_3 = \frac{1}{j} \), and \( V_{-3} = -\frac{1}{j} \). Putting it all together,

\[ V_k = \begin{cases} 
1 & k = 2, k = -2 \\
\frac{1}{j} & k = 3 \\
-\frac{1}{j} & k = -3 \\
0 & \text{otherwise}
\end{cases} \]

Problem 2.2

Suppose that \( x(t) \) is a square wave with a period of \( T_0 \), and with the following definition:

\[ x(t) = \begin{cases} 
\frac{1}{2} & -\frac{T_0}{4} < t < \frac{T_0}{4} \\
-\frac{1}{2} & \frac{T_0}{4} < t < \frac{3T_0}{4} \\
0 & \text{otherwise}
\end{cases} \]

We showed in class that the Fourier coefficients of this square wave are

\[ X_k = \begin{cases} 
0 & k \text{ even} \\
\frac{\left(-1\right)^{\left|k\right| - 1}/2}{\pi k} & k \text{ odd}
\end{cases} \]

Remember that the real-valued coefficients correspond to a Fourier series where all of the components are cosines. That makes sense, because the signal has even symmetry \( (x(t) = x(-t)) \), just like a cosine.
Suppose that we delay the signal by one quarter period, to produce the signal

\[ y(t) = \begin{cases} \frac{1}{2} & 0 < t < \frac{T_0}{2} \\ -\frac{1}{2} & \frac{T_0}{2} < t < T_0 \end{cases} \]

Notice that \( y(t) \) has odd symmetry \((y(t) = -y(-t))\), just like a sine wave. In that case, we might speculate that the Fourier series expansion will be composed entirely of sine waves, i.e., the Fourier series coefficients, \( Y_k \), will all be imaginary numbers.

Use the time-delay property of the spectrum (from lecture 3) to find out what happens to \( X_k \) when \( x(t) \) is delayed by exactly one quarter period.

**Solution:** The time-delay property of a spectrum is that if \( y(t) = x(t - \tau) \), then

\[ Y_k = X_k e^{-j2\pi f_k \tau} \]

where, in our case, \( f_k = kF_0 = k/T_0 \), and \( \tau = T_0/4 \), so

\[ Y_k = X_k e^{-j2\pi(k/T_0)(T_0/4)} = X_k e^{-j\pi k/2} = X_k \times (-j)^k \]

The multiplier \((-j)^k\) is imaginary for odd values of \( k \), and real for even values of \( j \). But we already know that \( X_k = 0 \) for even values of \( j \), so this time delay will make all of its nonzero coefficients into imaginary numbers. Specifically,

\[ Y_k = \begin{cases} 0 & k \text{ even} \\ \frac{(-1)^{(k+1)/2}}{\pi k} & k \text{ odd} \end{cases} \]

**Problem 2.3**

Suppose that \( z(t) \) is a triangle wave with a period of \( T_0 \), and with the following definition:

\[ z(t) = \begin{cases} \frac{4}{T_0} - \frac{t}{T_0} & 0 < t < \frac{T_0}{2} \\ \frac{4}{T_0} + \frac{t}{T_0} & \frac{T_0}{2} < t < T_0 \end{cases} \]

Notice that this signal is exactly the anti-derivative of the signal \( y(t) \) from problem (1), i.e., \( y(t) = dz/dt \). Use the differentiation property of the spectrum (from lecture 3) in order to find the Fourier series coefficients \( Z_k \).

**Solution:** The differentiation property of the spectrum says that, if \( y(t) = dz/dt \), then

\[ Y_k = j2\pi f_k Z_k \]

where, in our case, \( f_k = kF_0 = k/T_0 \), so \( Y_k = j2\pi kZ_k/T_0 \), so

\[ Z_k = \frac{T_0 Y_k}{P} j2\pi k = \begin{cases} 0 & k \text{ even} \\ \frac{(-1)^{(k-1)/2}}{2\pi^2 k^2} & k \text{ odd} \end{cases} \]

Notice that the \( Z_k \) are real numbers! If you plot \( z(t) \) as a function of time, you can see that it has even symmetry \((z(t) = z(-t))\).

**Problem 2.4**

Suppose that a violin is playing the note A4 (440Hz), but our recording quality is bad, so we only get the first two harmonics:

\[ x(t) = \sum_{k=-2}^{2} a_k e^{j2\pi k440t} \]
Suppose we measure the spectrum, and find it to be

\[ \{(-880, 0.01), (-440, 1), (0, 0), (440, 1), (880, 0.01)\} \]

In order to improve the balance a little, we try differentiating the tone. Our differentiator also imposes a delay and a DC offset, though, so what we get is

\[ y(t) = \frac{dx(t - 0.001)}{dt} + 1.5 \]

Find the spectrum of \( y(t) \).

**Solution:** The time delay multiplies each coefficient by a phase offset term, \( e^{-j2\pi f_k \tau} \). Differentiation multiplies each coefficient by \( j2\pi f_k \). The DC offset just adds 1.5 to the zeroth spectral coefficient. The resulting spectrum is

\[ \{(-880, -j2\pi8.8e^{j2\pi0.88}), (-440, -j2\pi440e^{j2\pi0.44}), (0, 1.5), (440, j2\pi440e^{-j2\pi0.44}), (880, j2\pi8.8e^{-j2\pi0.88})\} \]

which can be written as

\[ \{(-880, 2\pi8.8e^{j(2\pi0.88-\frac{\pi}{2})}), (-440, 2\pi440e^{j(2\pi0.44-\frac{\pi}{2})}), (0, 1.5), (440, 2\pi440e^{-j(2\pi0.44-\frac{\pi}{2})}), (880, 2\pi8.8e^{-j(2\pi0.88-\frac{\pi}{2})})\} \]

or equivalently

\[ \{(-880, 17.6\pi e^{j1.26\pi}), (-440, 880\pi e^{j0.38\pi0.44}), (0, 1.5), (440, 880\pi e^{-j0.38\pi}), (880, 17.6\pi e^{-j1.26\pi})\} \]