Problem 3.1

In MP3, one of the filters you’ll create is a local averaging filter. A local averaging filter produces an output \( y[n] \), at time \( n \), which is the average of the previous \( N \) samples of \( x[m] \):

\[
y[n] = \sum_{m=-\infty}^{\infty} h[m] x[n - m]
\]

(3.1-1)

\[
h[m] = \begin{cases} 
  \frac{1}{N} & 0 \leq m \leq N - 1 \\
  0 & \text{otherwise}
\end{cases}
\]

(3.1-2)

(a) First, consider what happens if \( x[m] \) is a pure tone with a period of \( N_0 = \frac{2\pi}{\omega_0} \), an amplitude of \( A \), and a phase of \( \theta \):

\[
x[n] = A \cos (\omega_0 n - \theta)
\]

Suppose that the averaging window, \( N \), is exactly an integer multiple of \( N_0 \). For example, suppose that \( N = 3N_0 \). Draw a picture of \( x[n] \) as a function of \( n \), and shade in the regions that would be added together by the summation in Eq. (3.1-1) in order to compute \( y[0] \). Argue based on your figure (with no calculations at all) that \( y[0] = 0 \).

(b) Adding up the samples of a cosine is easy when \( N \) is an integer multiple of \( N_0 \), but hard otherwise. It’s actually much easier to add the samples of a complex exponential, because we can use the standard geometric series formula (https://en.wikipedia.org/wiki/Geometric_series#Formula). Use that formula to find \( y[N - 1] \) when

\[
x[n] = e^{j\omega_0 n}
\]

Your result should have the form \( y[0] = (1 - a)/(1 - b) \) for some complex-valued constants, \( a \) and \( b \), that depend on \( \pi, N \), and \( \omega_0 \), but not on \( m \) or \( n \).

Problem 3.2

Another of the filters you’ll create in MP3 is a backward-difference filter. A backward-difference filter is one of several different ways of approximating a first-derivative:

\[
y[n] = \sum_{m=-\infty}^{\infty} h[m] x[n - m]
\]

(3.2-1)

\[
h[m] = \begin{cases} 
  1 & m = 0 \\
  -1 & m = 1 \\
  0 & \text{otherwise}
\end{cases}
\]

(3.2-2)
(a) First, consider what happens if $x[m]$ is a pure tone with a period of $T_0 = \frac{2\pi}{\omega_0}$, an amplitude of $A$, and a phase of $\theta$:

$$x[n] = A \cos (\omega_0 n - \theta)$$

Plug Eq. (3.2-3) into Eq. (3.2-1), then use the following trigonometric identity and the following approximation to discover in exactly what way the first-difference operator approximates a derivative:

$$-2 \sin(a) \sin(b) = \cos(a + b) - \cos(a - b)$$
$$\sin(b) \approx b \text{ if } b \text{ is small}$$

In order to apply the approximation, you can assume that $\omega_0$ is a small number.

(b) Let’s try the same thing with a complex exponential. Plug the following value of $x[n]$ into Eq. (3.2-1)

$$x[n] = e^{j\omega_0 n},$$

then assume that $\omega_0$ is a small number, and simplify using the approximation

$$e^{\phi} \approx 1 + \phi \text{ if } \phi \text{ is small}$$

in order to get something that looks like $dx[n]/dn$. 