This is a CLOSED BOOK exam.

You are permitted one sheet of handwritten notes, 8.5x11.

Calculators and computers are not permitted.

Do not simplify explicit numerical expressions. The expression \( e^{-5} \cos(3) \) is a MUCH better answer than \( -0.00667 \).

If you’re taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.

There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.

You must SHOW YOUR WORK to get full credit.

Name: __________________________
Convolution

\[ h[n] \ast x[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \]

Frequency Response

\[ H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \]

\[ h[n] \ast \cos(\omega n) = |H(\omega)| \cos(\omega n + \angle H(\omega)) \]

Rectangular Window and Ideal LPF

\[ w_R[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = e^{-\frac{j\pi(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)} \]

\[ h_{\text{ideal}}[n] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \leftrightarrow H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases} \]

Hamming Window

\[ w_H[n] = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \]

\[ W_H(\omega) = 0.54 W_R(\omega) + 0.23W_R\left(\omega - \frac{2\pi}{N-1}\right) + 0.23W_R\left(\omega + \frac{2\pi}{N-1}\right) \]
1. (25 points) Consider the following system:

\[ y[n] = \begin{cases} 
  x[n] & \text{if } n \text{ is even} \\
  0 & \text{if } n \text{ is odd} 
\end{cases} \]

(a) Is this system linear? Prove your answer.

**Solution:**

\[
x_1[n] \rightarrow y_1[n] = \begin{cases} 
  x_1[n] & \text{if } n \text{ is even} \\
  0 & \text{if } n \text{ is odd} 
\end{cases}
\]

\[
x_2[n] \rightarrow y_2[n] = \begin{cases} 
  x_2[n] & \text{if } n \text{ is even} \\
  0 & \text{if } n \text{ is odd} 
\end{cases}
\]

\[ x_3[n] = x_1[n] + x_2[n] \]

\[
x_3[n] \rightarrow y_3[n] = \begin{cases} 
  x_3[n] & \text{if } n \text{ is even} \\
  0 & \text{if } n \text{ is odd} 
\end{cases}
\]

\[ = \begin{cases} 
  x_1[n] + x_2[n] & \text{if } n \text{ is even} \\
  0 & \text{if } n \text{ is odd} 
\end{cases} \]

\[ = y_1[n] + y_2[n] \]

So yes, it is linear.

(b) Is this system shift-invariant? Prove your answer.

**Solution:**

\[
x_1[n] \rightarrow y_1[n] = \begin{cases} 
  x_1[n] & \text{if } n \text{ is even} \\
  0 & \text{if } n \text{ is odd} 
\end{cases}
\]

\[ x_2[n] = x_1[n - 1] \]

\[
x_2[n] \rightarrow y_2[n] = \begin{cases} 
  x_2[n] & \text{if } n \text{ is even} \\
  0 & \text{if } n \text{ is odd} 
\end{cases}
\]

\[ = \begin{cases} 
  x_1[n - 1] & \text{if } n \text{ is even} \\
  0 & \text{if } n \text{ is odd} 
\end{cases} \]

\[ \neq y_1[n - 1] \]

So it is NOT shift-invariant.
2. (25 points) The second-derivative of a sampled signal may be approximated by an LSI system with the following impulse response:

\[ h[n] = -\delta[n] + \frac{1}{2} (\delta[n + 1] + \delta[n - 1]) \]

(a) Is this system causal? Why or why not?

**Solution:** No, because \( h[n] \) is not right-sided.

(b) Is this system stable? Why or why not?

**Solution:** Yes, because \( \sum_{n=-\infty}^{\infty} |h[n]| = 2 \), which is finite.

(c) What is \( h[n] * h[n] \)? Give the time and value of every nonzero sample, using either a sketch or an equation.

**Solution:**

\[ h[n] * h[n] = 1.5\delta[n] - (\delta[n - 1] + \delta[n + 1]) + 0.25 (\delta[n - 2] + \delta[n + 2]) \]
3. (25 points) Consider, again, the same LSI system that was used in problem 2. Remember that its impulse response is

\[ h[n] = -\delta[n] + \frac{1}{2} (\delta[n + 1] + \delta[n - 1]) \]

(a) What is its frequency response?

Solution:

\[ H(\omega) = -1 + \frac{1}{2} (e^{j\omega} + e^{-j\omega}) \]
\[ = -1 + \cos(\omega) \]

(b) Consider a system \( G \) that accepts as input a signal \( x[n] \), and generates as output a signal \( z[n] \). The system \( G \) does two things to \( x[n] \). First, it convolves it with \( h[n] \), producing \( y[n] = x[n] * h[n] \). Second, it delays it by five samples, producing \( z[n] = y[n - 5] \). What is the impulse response of the system \( G \)?

Solution:

\[ g[n] = -\delta[n - 5] + \frac{1}{2} (\delta[n - 4] + \delta[n - 6]) \]
4. (25 points) You have an application for which it is necessary to increase the amplitude of inputs below \( \frac{\pi}{4} \) radians/sample, while eliminating inputs above \( \frac{3\pi}{4} \) radians/sample. The ideal filter is therefore:

\[
H_i(\omega) = \begin{cases} 
2 & |\omega| < \frac{\pi}{4} \\
1 & \frac{\pi}{4} < |\omega| < \frac{3\pi}{4} \\
0 & \frac{3\pi}{4} < |\omega| < \pi 
\end{cases}
\]

(a) What is \( h_i[n] \)?

**Solution:** There are several ways to write it. One is:

\[
h_i[n] = \frac{3}{4} \text{sinc} \left( \frac{3\pi n}{4} \right) + \frac{1}{4} \text{sinc} \left( \frac{\pi n}{4} \right)
\]

(b) You propose to create a realizable filter, \( h[n] \), by windowing \( h_i[n] \) with a length-127 Hamming window: \( h[n] = w[n]h_i[n] \). In the resulting frequency response, \( H(\omega) \), how wide are the transition bands?

**Solution:** \( \frac{3\pi}{127} \) radians/sample.