UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Spring 2022

EXAM 2

Monday, October 31, 2022

- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

Convolution

$$h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Frequency Response

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$h[n] * \cos(\omega n) = |H(\omega)| \cos(\omega n + \angle H(\omega))$$

Rectangular Window and Ideal LPF

$$w_R[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = e^{-\frac{j\omega(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$h_{\text{ideal}}[n] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \leftrightarrow H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Hamming Window

$$w_H[n] = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$W_H(\omega) = 0.54 W_R(\omega) + 0.23 W_R\left(\omega - \frac{2\pi}{N-1}\right) + 0.23 W_R\left(\omega + \frac{2\pi}{N-1}\right)$$

1. (25 points) Consider the following system:

$$y[n] = \begin{cases} x[n] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

(a) Is this system linear? Prove your answer.

Solution:

$$\begin{aligned}
x_1[n] \rightarrow y_1[n] &= \begin{cases} x_1[n] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\
x_2[n] \rightarrow y_2[n] &= \begin{cases} x_2[n] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\
x_3[n] &= x_1[n] + x_2[n] \\
x_3[n] \rightarrow y_3[n] &= \begin{cases} x_3[n] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} x_1[n] + x_2[n] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\
&= y_1[n] + y_2[n]
\end{aligned}$$

So yes, it is linear.

(b) Is this system shift-invariant? Prove your answer.

Solution:

$$\begin{aligned}
x_1[n] \rightarrow y_1[n] &= \begin{cases} x_1[n] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\
x_2[n] &= x_1[n-1] \\
x_2[n] \rightarrow y_2[n] &= \begin{cases} x_2[n] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} x_1[n-1] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\
&\neq y_1[n-1]
\end{aligned}$$

So it is NOT shift-invariant.

2. (25 points) The second-derivative of a sampled signal may be approximated by an LSI system with the following impulse response:

$$h[n] = -\delta[n] + \frac{1}{2}\left(\delta[n+1] + \delta[n-1]\right)$$

(a) Is this system causal? Why or why not?

Solution: No, because $h[n]$ is not right-sided.

(b) Is this system stable? Why or why not?

Solution: Yes, because $\sum_{n=-\infty}^{\infty} |h[n]| = 2$, which is finite.

(c) What is $h[n] * h[n]$? Give the time and value of every nonzero sample, using either a sketch or an equation.

Solution:

$$h[n] * h[n] = 1.5\delta[n] - \left(\delta[n-1] + \delta[n+1]\right) + 0.25\left(\delta[n-2] + \delta[n+2]\right)$$

3. (25 points) Consider, again, the same LSI system that was used in problem 2. Remember that its impulse response is

$$h[n] = -\delta[n] + \frac{1}{2}\left(\delta[n+1] + \delta[n-1]\right)$$

(a) What is its frequency response?

Solution:

$$\begin{aligned} H(\omega) &= -1 + \frac{1}{2}\left(e^{j\omega} + e^{-j\omega}\right) \\ &= -1 + \cos(\omega) \end{aligned}$$

(b) Consider a system $\mathcal{G}$ that accepts as input a signal $x[n]$, and generates as output a signal $z[n]$. The system $\mathcal{G}$ does two things to $x[n]$. First, it convolves it with $h[n]$, producing $y[n] = x[n] * h[n]$. Second, it delays it by five samples, producing $z[n] = y[n-5]$. What is the impulse response of the system $\mathcal{G}$?

Solution:

$$g[n] = -\delta[n-5] + \frac{1}{2}\left(\delta[n-4] + \delta[n-6]\right)$$

4. (25 points) You have an application for which it is necessary to increase the amplitude of inputs below $\frac{\pi}{4}$ radians/sample, while eliminating inputs above $\frac{3\pi}{4}$ radians/sample. The ideal filter is therefore:

$$H_i(\omega) = \begin{cases} 2 & |\omega| < \frac{\pi}{4} \\ 1 & \frac{\pi}{4} < |\omega| < \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} < |\omega| < \pi \end{cases}$$

(a) What is $h_i[n]$?

Solution: There are several ways to write it. One is:

$$h_i[n] = \frac{3}{4}\text{sinc}\left(\frac{3\pi n}{4}\right) + \frac{1}{4}\text{sinc}\left(\frac{\pi n}{4}\right)$$

(b) You propose to create a realizable filter, $h[n]$, by windowing $h_i[n]$ with a length-127 Hamming window: $h[n] = w[n]h_i[n]$. In the resulting frequency response, $H(\omega)$, how wide are the transition bands?

Solution: $\frac{8\pi}{127}$ radians/sample.

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