• This is a CLOSED BOOK exam.
• You are permitted one sheet of handwritten notes, 8.5x11.
• Calculators and computers are not permitted.
• If you’re taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
• There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
• You must SHOW YOUR WORK to get full credit.
Phasors

\[ A \cos(2\pi ft + \theta) = \Re \{ Ae^{j\theta} e^{j2\pi ft} \} = \frac{1}{2} e^{-j\theta} e^{-j2\pi ft} + \frac{1}{2} e^{j\theta} e^{j2\pi ft} \]

Spectrum

Scaling: \( y(t) = Gx(t) = \sum_{k=-N}^{N} (G_{ak}) e^{j2\pi f_k t} \)

Add a Constant: \( y(t) = x(t) + C = (a_0 + C) + \sum_{k \neq 0} a_k e^{j2\pi f_k t} \)

Add Signals: If \( f_k = f'_n = f''_m \) then \( a_k = a'_n + a''_m \)

Time Shift: \( y(t) = x(t - \tau) = \sum_{k=-N}^{N} \left( a_k e^{-j2\pi f_k \tau} \right) e^{j2\pi f_k t} \)

Frequency Shift: \( y(t) = x(t)e^{j2\pi Ft} = \sum_{k=-N}^{N} a_k e^{j2\pi (f_k + F)t} \)

Differentiation: \( y(t) = \frac{dx}{dt} = \sum_{k=-N}^{N} (j2\pi f_k a_k) e^{j2\pi f_k t} \)

Fourier Series

Analysis: \( X_k = \frac{1}{T_0} \int_{0}^{T_0} x(t)e^{-j2\pi k t/T_0} dt \)

Synthesis: \( x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k t/T_0} \)

Sampling and Interpolation:

\[ x[n] = x \left( t = \frac{n}{F_s} \right) \]

\( f_a = \min (f \mod F_s, -f \mod F_s) \)

\( z_a = \begin{cases} z & f \mod F_s < -f \mod F_s \\ z^* & f \mod F_s > -f \mod F_s \end{cases} \)

\( y(t) = \sum_{n=-\infty}^{\infty} y[n] p(t - nT_s) \)
1. (25 points) Suppose that

\[ x(t) = -12 \cos \left(1000\pi t - \frac{\pi}{4}\right) + 4 \sin (1000\pi t) = M \cos (1000\pi t + \theta) \]

Find \( x \) and \( y \) such that \( M = \sqrt{x^2 + y^2} \) and either \( \theta = \tan(y/x) \) or \( \theta = \tan(y/x) - \pi \).

Solution:

\[
\begin{align*}
x &= -12 \cos \left(-\frac{\pi}{4}\right) = -6\sqrt{2} \\
y &= -12 \sin \left(-\frac{\pi}{4}\right) - 4 = 6\sqrt{2} - 4
\end{align*}
\]
2. (25 points) $x(t)$ is a signal with a period of 0.01 seconds, and with the following shape:

$$x(t) = \begin{cases} 
1 & 0 < t < 0.001 \\
0 & 0.001 < t < 0.005 \\
-1 & 0.005 < t < 0.006 \\
0 & 0.006 < t < 0.01 
\end{cases}$$

(a) What are the Fourier series coefficients $X_k$ for $k \neq 0$? Your answer should contain no variables other than $k$, but you don’t need to simplify.

**Solution:**

$$X_k = \frac{1}{-j2\pi k} \left( e^{-j\frac{2\pi \cdot 0.001}{0.01}} - 1 \right) - \frac{1}{-j2\pi k} \left( e^{-j\frac{2\pi \cdot 0.006}{0.01}} - e^{-j\frac{2\pi \cdot 0.005}{0.01}} \right)$$

$$= \frac{j}{2\pi k} \left( e^{-j\frac{\pi k}{5}} - 1 - e^{-j\frac{6\pi k}{5}} + e^{-j\pi k} \right)$$

... and this can be further simplified (a lot), but let’s leave it here for now.
(b) Suppose that $y(t)$ is a signal such that $x(t) = \frac{dy}{dt}$. Express the Fourier series coefficients $Y_k$ in terms of the Fourier series coefficients $X_k$. Note that you don’t need to solve part (a) in order to solve this part of the problem.

**Solution:**

$$y(t) = \sum Y_k e^{j2\pi kt/T_0}$$

$$\frac{dy}{dt} = \sum \frac{j2\pi k}{0.01} Y_k e^{j2\pi kt/T_0}$$

...so therefore,

$$Y_k = \frac{0.01}{j2\pi k} X_k$$
3. (25 points) Suppose $x(t)$ is sampled to create the signal $y[n] = x \left( \frac{n}{F_s} \right)$, with a sampling frequency of $F_s = 10,000$ samples/second. The signal $y[n]$ is then passed through an ideal D/A in order to produce the signal $z(t)$. In each of the two following cases, what is $z(t)$?

(a) $x(t) = 3 \cos \left( 2\pi 8000t + \frac{\pi}{4} \right)$

What is $z(t)$?

**Solution:** $f_a = F_s - f = 2000$Hz, so $z_a = z^* = 3e^{-j\pi/4}$, and

$$z(t) = 3 \cos \left( 2\pi 2000t - \frac{\pi}{4} \right)$$
(b) 

\[ x(t) = 3 \cos \left( 2\pi 12,000 t + \frac{\pi}{4} \right) \]

What is \( z(t) \)?

**Solution:** \( f_a = f - F_a = 2000 \text{Hz} \), so \( z_a = z = 3e^{j\pi/4} \), and

\[ z(t) = 3 \cos \left( 2\pi 2000 t + \frac{\pi}{4} \right) \]
4. (25 points) Suppose that

\[ x[n] = \sin \left( \frac{\pi n}{2} \right) = \begin{cases} 1 & \text{n is odd, and } \frac{n-1}{2} \text{ is even} \\ -1 & \text{n is odd, and } \frac{n-1}{2} \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \]

You would like to generate a continuous-time audio signal, \( y(t) \), using the interpolation formula

\[ y(t) = \sum_{n=-\infty}^{\infty} x[n] g(t-n) \]

In each of the following cases, specify the value of \( y(t) \) over the range \( 0 \leq t \leq 4 \). You may specify \( y(t) \) by drawing a plot of the function (if your plot clearly shows the value at each point in time in the range \( 0 \leq t \leq 4 \)), or by using an equation or a set of cases.

(a) What is \( y(t) \) if

\[ g(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \]

\[ y(t) = \begin{cases} 1 & \frac{1}{2} \leq t \leq \frac{3}{2} \\ -1 & \frac{5}{2} \leq t \leq \frac{7}{2} \\ 0 & \text{otherwise} \end{cases} \]
(b) What is $y(t)$ if

$$g(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

**Solution:**

$$y(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2 - t & 1 \leq t \leq 3 \\ t - 4 & 3 \leq t \leq 4 \end{cases}$$

(c) What is $y(t)$ if

$$g(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & \text{otherwise} \end{cases}$$

**Solution:**

$$y(t) = \sin \left( \frac{\pi t}{2} \right)$$