• This is a CLOSED BOOK exam. You may use three pages (front and back) of your own notes, and you may use a calculator if you wish.

• There are a total of 200 points in the exam. Each problem specifies its point total. Plan your work accordingly.

• You must SHOW YOUR WORK to get full credit.

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Useful Angles

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<tr>
<th>$\theta$</th>
<th>$\cos \theta$</th>
<th>$\sin \theta$</th>
<th>$e^{j\theta}$</th>
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<tr>
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<td>-1</td>
</tr>
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<td>1</td>
<td>-1</td>
<td>$-j$</td>
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<tr>
<td>$2\pi$</td>
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<td>0</td>
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Useful DTFTs

$$x[n] = a^n u[n] \leftrightarrow X(\omega) = \frac{1}{1 - az^{-1}}$$
$$x[n] = \delta[n - k] \leftrightarrow X(\omega) = e^{-j\omega k}$$
$$x[n] = e^{j\theta n} \leftrightarrow X(\omega) = 2\pi \delta(\omega - \theta)$$
$$x[n] = \left(\frac{\omega_c}{\pi}\right) \text{sinc}(\omega_c n) \leftrightarrow X(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$
$$x[n] = \begin{cases} 1 & |n| \leq M \\ 0 & \text{otherwise} \end{cases} \leftrightarrow X(\omega) = \frac{\sin(\omega(2M + 1)/2)}{\sin(\omega/2)}$$
Problem 1  (20 points)

\[ 6 \cos \left( 2\pi \frac{1000}{4000} \left( t - \frac{1}{4000} \right) \right) + 6 \sin \left( 2\pi \frac{1000}{4000} \left( t - \frac{1}{4000} \right) \right) = A \cos(\Omega t + \phi) \]

Find the following quantities:

Solution:

\[
\begin{align*}
A &= 6\sqrt{2} \\
\Omega &= \frac{2\pi}{4000} \\
\phi &= -\frac{3\pi}{4}
\end{align*}
\]
Problem 2  (20 points)

A periodic signal $x(t)$, with period $T_0$, is given by

$$ x(t) = \begin{cases} 
1 & 0 \leq t \leq \frac{3T_0}{4} \\
0 & \frac{3T_0}{4} < t < T_0 
\end{cases} $$

The same signal can be expressed as a Fourier series:

$$ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} $$

Find $|X_2|$, the amplitude of the second harmonic.

Solution:

$$ |X_2| = \frac{1}{2\pi} $$
Problem 3  (20 points)

A particular system generates an output $y[n]$ from its input $x[n]$ according to the following rule:

$$y[n] = \begin{cases} 
    x[n] & n \text{ is even} \\
    \frac{1}{2} (x[n - 1] + x[n + 1]) & n \text{ is odd}
\end{cases}$$

(a) (6 points) Is the system linear? Give your reason.

(b) (4 points) Is the system causal? Give your reason.
(c) (6 points) Is the system time-invariant? Give your reason.

(d) (4 points) Is the system stable? Give your reason.
Problem 4  (20 points)

Find $y[n] = h[n] * x[n]$, where

$$x[n] = \cos (0.02\pi n), \quad h[n] = \begin{cases} 1 & |n| \leq 3 \\ 0 & |n| > 3 \end{cases}$$

What is $y[n]$? Hint: Find $H(\omega)$ first. In order to find the numerical value of your answer, you may find it useful to approximate $\sin x \approx x$, an approximation that works for small values of $x$. 
Problem 5  (20 points)

Find \( y[n] = h[n] \ast x[n] \), where

\[
x[n] = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0
\end{cases}, \quad h[n] = \begin{cases} 
1 & |n| \leq 3 \\
0 & |n| > 3
\end{cases}
\]

What is \( y[n] \)?
Problem 6  (20 points)

Suppose

\[ x[n] = \cos \left( \frac{7\pi n}{21} \right), \quad y[n] = \begin{cases} x[n] & |n| \leq 10 \\ 0 & \text{otherwise} \end{cases}, \quad Y(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} \]

Sketch \( Y(\omega) \) for \(-\pi \leq \omega \leq \pi\). Specify the frequency and amplitude of at least one peak. Also, specify at least three particular frequencies \( \omega \) such that \( Y(\omega) = 0 \).
Problem 7  (20 points)

You have a 250 × 250 image that you want to upsample to 250 × 1000 without introducing any aliasing. If \( x[n] \) is a row of the original image, and \( y[n] \) is a row of the upsampled image, this task can be accomplished by

\[
y[n] = \sum_{m=0}^{249} x[m]g[n - 4m]
\]

Sketch \( g[n] \) as a function of \( n \). Show the value of \( g[0] \), and specify at least three particular sample indices, \( n \), at which \( g[n] = 0 \).
Problem 8  (20 points)

An 8000Hz tone, \(x(t) = \cos(2\pi 8000t)\), is sampled at \(F_s = \frac{1}{T} = 10,000\) samples/second in order to create \(x[n] = x(nT)\). Sketch \(X(\omega)\) for \(0 \leq \omega \leq 2\pi\) (note the domain!!). Specify the frequencies at which \(X(\omega) \neq 0\).

Solution:

\(X(\omega)\) is a spectrum with energy at the frequencies \((0.4\pi, 1.6\pi)\).
Problem 9  (20 points)

Suppose \( x[n] \) is a random signal with the following autocorrelation:

\[
R_{xx}[\tau] = \frac{1}{16} \sin^2 \left( \frac{\pi n}{4} \right) = \left( \frac{\sin(\pi n/4)}{\pi n} \right)^2
\]

Suppose \( e[n] = x[n] - ax[n-1] \), and you want to find \( a \) in order to minimize \( E[e^2[n]] \). Find the numerical value of \( a \) ("numerical" in the sense that there are no variables in your answer, however, your answer may include constants like \( \pi \) and \( \sqrt{2} \)).
Problem 10  (20 points)

Suppose \( y[n] = x[n] + v[n] \). \( v[n] \) is zero-mean, unit-variance white noise uncorrelated with \( x[n] \), and \( x[n] \) is a random signal whose power spectrum is given by

\[
P_{xx}(\omega) = \begin{cases} 
\frac{\pi}{2} - |\omega| & |\omega| \leq \frac{\pi}{2} \\
0 & \frac{\pi}{2} \leq |\omega| \leq \pi 
\end{cases}
\]

Suppose \( z[n] = h[n] * y[n] \). Find \( H(\omega) \) in order to minimize \( E \left[ (z[n] - x[n])^2 \right] \).