ECE 401: Signal and Image Analysis, Fall 2021

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1. Review: DTFT
2. DFT
3. Example
4. Example: Shifted Delta Function
5. Example: Cosine
6. Properties of the DFT
7. Summary
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The DTFT (discrete time Fourier transform) of any signal is $X(\omega)$, given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n - n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$
Properties of the DTFT

Properties worth knowing include:

1. **Periodicity:** \( X(\omega + 2\pi) = X(\omega) \)

2. **Linearity:**
   \[
   z[n] = ax[n] + by[n] \iff Z(\omega) = aX(\omega) + bY(\omega)
   \]

3. **Time Shift:** \( x[n - n_0] \iff e^{-jn_0}\omega X(\omega) \)

4. **Frequency Shift:** \( e^{jn_0\omega}x[n] \iff X(\omega - \omega_0) \)

5. **Filtering is Convolution:**
   \[
   y[n] = h[n] * x[n] \iff Y(\omega) = H(\omega)X(\omega)
   \]
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How can we compute the DTFT?

- The DTFT has a big problem: it requires an infinite-length summation, therefore you can’t compute it on a computer.
- The DFT solves this problem by assuming a finite length signal.
- “$N$ equations in $N$ unknowns:” if there are $N$ samples in the time domain ($x[n], \ 0 \leq n \leq N - 1$), then there are only $N$ independent samples in the frequency domain ($X(\omega_k), \ 0 \leq k \leq N - 1$).
First, assume that $x[n]$ is nonzero only for $0 \leq n \leq N - 1$. Then the DTFT can be computed as:

$$X(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$
N equations in N unknowns

Since there are only $N$ samples in the time domain, there are also only $N$ independent samples in the frequency domain:

$$X[k] = X(\omega_k) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

where

$$\omega_k = \frac{2\pi k}{N}, \quad 0 \leq k \leq N - 1$$
Putting it all together, we get the formula for the DFT:

\[ X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi kn}{N}} \]
Inverse Discrete Fourier Transform

\[ X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi kn}{N}} \]

Using orthogonality, we can also show that

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi kn}{N}} \]
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Consider the signal

\[ x[n] = \begin{cases} 
  1 & \text{if } n=0,1 \\
  0 & \text{if } n=2,3 \\
  \text{undefined} & \text{otherwise}
\end{cases} \]
Example DFT

\[ X[k] = \sum_{n=0}^{3} x[n] e^{-j \frac{2\pi kn}{4}} \]

\[ = 1 + e^{-j \frac{2\pi k}{4}} \]

\[ = \begin{cases} 
2 & k = 0 \\
1 - j & k = 1 \\
0 & k = 2 \\
1 + j & k = 3 
\end{cases} \]
Example IDFT

\[ X[k] = [2, (1 - j), 0, (1 + j)] \]

\[
x[n] = \frac{1}{4} \sum_{k=0}^{3} X[k] e^{j \frac{2\pi kn}{4}}
\]

\[
= \frac{1}{4} \left( 2 + (1 - j)e^{j \frac{2\pi n}{4}} + (1 + j)e^{j \frac{6\pi n}{4}} \right)
\]

\[
= \frac{1}{4} \left( 2 + (1 - j)j^n + (1 + j)(-j)^n \right)
\]

\[
= \begin{cases} 
1 & n = 0, 1 \\
0 & n = 2, 3
\end{cases}
\]
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In many cases, we can find the DFT directly from the DTFT. For example:

\[ h[n] = \delta[n - n_0] \quad \leftrightarrow \quad H(\omega) = e^{-j\omega n_0} \]

**If and only if the signal is less than length** \( N \), **we can just plug in** \( \omega_k = \frac{2\pi k}{N} \):

\[ h[n] = \delta[n - n_0] \quad \leftrightarrow \quad H[k] = \begin{cases} e^{-j\frac{2\pi kn_0}{N}} & 0 \leq n_0 \leq N - 1 \\ \text{undefined} & \text{otherwise} \end{cases} \]
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Finding the DFT of a cosine is possible, but harder than you might think. Consider:

\[ x[n] = \cos(\omega_0 n) \]

This signal violates the first requirement of a DFT:

- \( x[n] \) must be finite length.
We can make $x[n]$ finite-length by windowing it, like this:

$$x[n] = \cos(\omega_0 n) w[n],$$

where $w[n]$ is the rectangular window,

$$w[n] = \begin{cases} 
1 & 0 \leq n \leq N - 1 \\
0 & \text{otherwise}
\end{cases}$$
Now that $x[n]$ is finite length, we can just take its DTFT, and then sample at $\omega_k = \frac{2\pi k}{N}$:

$$X[k] = X(\omega_k) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n}$$
But how do we solve this equation?

\[ X(\omega_k) = \sum_{n=0}^{N-1} \cos(\omega_0 n) w[n] e^{-j\omega_k n} \]

The answer is, surprisingly, that we can use two properties of the DTFT:

- **Linearity:** \( x_1[n] + x_2[n] \iff X_1(\omega) + X_2(\omega) \)
- **Frequency Shift:** \( e^{j\omega_0 n} z[n] \iff Z(\omega - \omega_0) \)
**Linearity and Frequency-Shift Properties of the DTFT**

- **Linearity:**

  \[
  \cos(\omega_0 n)w[n] = \frac{1}{2}e^{j\omega_0 n}w[n] + \frac{1}{2}e^{-j\omega_0 n}w[n]
  \]

- **Frequency Shift:**

  \[
  e^{j\omega_0 n}w[n] \leftrightarrow W(\omega - \omega_0)
  \]

Putting them together, we have that

\[
\cos(\omega_0 n)w[n] \leftrightarrow \frac{1}{2}W(\omega - \omega_0) + \frac{1}{2}W(\omega + \omega_0)
\]
DFT of a Cosine

Putting it together,

\[ x[n] = \cos(\omega_0 n)w[n] \leftrightarrow X(\omega_k) = \frac{1}{2} W(\omega_k - \omega_0) + \frac{1}{2} W(\omega_k + \omega_0) \]

where \( W(\omega) \) is the Dirichlet form:

\[ W(\text{omega}) = e^{-j \omega \frac{N-1}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)} \]
Here’s the DFT of

\[ x[n] = \cos \left( \frac{2\pi 20.3}{N} \right) w[n] \]
DFT of a Cosine

Remember that $W(\omega) = 0$ whenever $\omega$ is a multiple of $\frac{2\pi}{N}$. But the DFT only samples at multiples of $\frac{2\pi}{N}$! So if $\omega_0$ is also a multiple of $\frac{2\pi}{N}$, then the DFT of a cosine is just a pair of impulses in frequency:
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Just as $X(\omega)$ is periodic with period $2\pi$, in the same way, $X[k]$ is periodic with period $N$:

$$X[k + N] = \sum_n x[n] e^{-j\frac{2\pi(k+N)n}{N}}$$

$$= \sum_n x[n] e^{-j\frac{2\pi kn}{N}} e^{-j\frac{2\pi Nn}{N}}$$

$$= \sum_n x[n] e^{-j\frac{2\pi kn}{N}}$$

$$= X[k]$$
The inverse DFT is also periodic in time! \( x[n] \) is undefined outside \( 0 \leq n \leq N - 1 \), but if we accidentally try to compute \( x[n] \) at any other times, we end up with:

\[
x[n + N] = \frac{1}{N} \sum_{k} X[k] e^{j \frac{2\pi k (n+N)}{N}}
\]

\[
= \frac{1}{N} \sum_{k} X[k] e^{j \frac{2\pi kn}{N}} e^{j \frac{2\pi kN}{N}}
\]

\[
= \frac{1}{N} \sum_{k} X[k] e^{j \frac{2\pi kn}{N}}
\]

\[
= x[n]
\]
Linearity

\[ ax_1[n] + bx_2[n] \leftrightarrow aX_1[k] + bX_2[k] \]
Samples of the DTFT

If \( x[n] \) is finite length, with length of at most \( N \) samples, then

\[
X[k] = X(\omega_k), \quad \omega_k = \frac{2\pi k}{N}
\]
Here’s a property of the DTFT that we didn’t talk about much. Suppose that $x[n]$ is real. Then

$$X(-\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(-\omega)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{j\omega n}$$

$$= \left( \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right)^*$$

$$= X^*(\omega)$$
Conjugate Symmetry of the DFT

\[ X(\omega) = X^*(-\omega) \]

Remember that the DFT, \( X[k] \), is just the samples of the DTFT, sampled at \( \omega_k = \frac{2\pi k}{N} \). So that means that conjugate symmetry also applies to the DFT:

\[ X[k] = X^*[-k] \]

But remember that the DFT is periodic with a period of \( N \), so

\[ X[k] = X^*[-k] = X^*[N - k] \]
The frequency shift property of the DTFT also applies to the DFT:

\[ w[n] e^{j\omega_0 n} \leftrightarrow W(\omega - \omega_0) \]

If \( \omega = \frac{2\pi k}{N} \), and if \( \omega_0 = \frac{2\pi k_0}{N} \), then we get

\[ w[n] e^{j\frac{2\pi k_0 n}{N}} \leftrightarrow W[k - k_0] \]
The time shift property of the DTFT was

\[ x[n - n_0] \leftrightarrow e^{j\omega n_0} X(\omega) \]

The same thing also applies to the DFT, except that the DFT is finite in time. Therefore we have to use what’s called a “circular shift:”

\[ x[((n - n_0))_N] \leftrightarrow e^{j\frac{2\pi kn_0}{N}} X[k] \]

where \(((n - n_0))_N\) means “\(n - n_0\), modulo \(N\).” We’ll talk more about what that means in the next lecture.
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### DFT Examples

1. \[ x[n] = [1, 1, 0, 0] \leftrightarrow X[k] = [2, 1 - j, 0, 1 + j] \]

2. \[ x[n] = \delta[n - n_0] \leftrightarrow X[k] = \begin{cases} e^{-j \frac{2\pi k n_0}{N}} & 0 \leq n_0 \leq N - 1 \\ \text{undefined} & \text{otherwise} \end{cases} \]

3. \[ x[n] = w[n] \cos(\omega_0 n) \leftrightarrow X[k] = \frac{1}{2} W \left[ k - \frac{N\omega_0}{2\pi} \right] + \frac{1}{2} W \left[ k + \frac{N\omega_0}{2\pi} \right] \]
DFT Properties

1. **Periodic in Time and Frequency:**
   \[ x[n] = x[n + N], \quad X[k] = X[k + N] \]

2. **Linearity:**
   \[ ax_1[n] + bx_2[n] \iff aX_1[k] + bX_2[k] \]

3. **Samples of the DTFT:** if \( x[n] \) has length at most \( N \) samples, then
   \[ X[k] = X(\omega_k), \quad \omega_k = \frac{2\pi k}{N} \]

4. **Frequency Shift:**
   \[ x[n] = e^{j \frac{2\pi k_0 n}{N}} \iff X[k - k_0] \]
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Show that the signal $x[n] = \delta[n - n_0]$ obeys the conjugate symmetry properties of both the DFT and DTFT.