Lecture 10: Ideal Filters

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ECE 401: Signal and Image Analysis, Fall 2020
1. Review: Ideal Filters

2. Realistic Filters: Finite Length

3. Realistic Filters: Even Length

4. Summary

5. Written Example
Outline

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### Review: Ideal Filters

- **Ideal Lowpass Filter:**

  \[ H_{LP}(\omega) = \begin{cases} 
  1 & |\omega| \leq \omega_c, \\
  0 & \omega_c < |\omega| \leq \pi. 
\end{cases} \quad \leftrightarrow \quad h_{LP}[m] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \]

- **Ideal Highpass Filter:**

  \[ H_{HP}(\omega) = 1 - H_{LP}(\omega) \quad \leftrightarrow \quad h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \]

- **Ideal Bandpass Filter:**

  \[ H_{BP}(\omega) = H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega) \]

  \[ \leftrightarrow h_{BP}[n] = \frac{\omega_2}{\pi} \text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi} \text{sinc}(\omega_1 n) \]
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Ideal Filters are Infinitely Long

- All of the ideal filters, $h_{LP,i}[n]$ and so on, are infinitely long!
- In demos so far, I’ve faked infinite length by just making $h_{LP,i}[n]$ more than twice as long as $x[n]$.
- If $x[n]$ is very long (say, a 24-hour audio recording), you probably don’t want to do that (computation=expensive)
Finite Length by Truncation

We can force $h_{LP,i}[n]$ to be finite length by just truncating it, say, to $2M + 1$ samples:

$$h_{LP}[n] = \begin{cases} h_{LP,i}[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$
The problem with truncation is that it causes artifacts.
Windowing Reduces the Artifacts

We can reduce the artifacts (a lot) by windowing $h_{LP,i}[n]$, instead of just truncating it:

$$h_{LP}[n] = \begin{cases} w[n]h_{LP,i}[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

where $w[n]$ is a window that tapers smoothly down to near zero at $n = \pm M$, e.g., a Hamming window:

$$w[n] = 0.54 + 0.46 \cos \left( \frac{2\pi n}{2M} \right)$$
Windowing a Lowpass Filter

Truncated $h_{LP}[n]$, cutoff=$\pi/4$

Hamming Window $w[n]$, Length=19

Windowed Filter $w[n]h_{LP}[n]$, Length=19
Windowing Reduces the Artifacts
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Even Length Filters

Often, we’d like our filter $h_{LP}[n]$ to be even length, e.g., 200 samples long, or 256 samples. We can’t do that with this definition:

$$h_{LP}[n] = \begin{cases} w[n]h_{LP,i}[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

...because $2M + 1$ is always an odd number.
We can solve this problem using the time-shift property of the DTFT:

$$z[n] = x[n - n_0] \iff Z(\omega) = e^{-j\omega n_0} X(\omega)$$
Even Length Filters using Delay

Let’s delay the ideal filter by exactly $M - 0.5$ samples, for any integer $M$:

$$z[n] = h_{LP,i} [n - (M - 0.5)] = \frac{\omega_c}{\pi} \text{sinc} \left( \omega \left( n - M + \frac{1}{2} \right) \right)$$

I know that sounds weird. But notice the symmetry it gives us. The whole signal is symmetric w.r.t. sample $n = M - 0.5$. So $z[M - 1] = z[M]$, and $z[M - 2] = z[M + 1]$, and so one, all the way out to

$$z[0] = z[2M - 1] = \frac{\omega_c}{\pi} \text{sinc} \left( \omega \left( M - \frac{1}{2} \right) \right)$$
Even Length Filters using Delay

Ideal LPF, delayed by 9.5 samples
Even Length Filters using Delay

Apply the time delay property:

\[ z[n] = h_{LP,i}[n - (M - 0.5)] \quad \leftrightarrow \quad Z(\omega) = e^{-j\omega(M-0.5)}H_{LP,i}(\omega), \]

and then notice that

\[ |e^{-j\omega(M-0.5)}| = 1 \]

So

\[ |Z(\omega)| = |H_{LP,i}(\omega)| \]
Even Length Filters using Delay

Ideal LPF, delayed by 9.5 samples

Magnitude of delayed filter $= |H_{LP}(\omega)|$

Phase of delayed filter $\angle H_{LP}(\omega) = -9.5\omega$
Even Length Filters using Delay and Windowing

Now we can create an even-length filter by windowing the delayed filter:

\[
  h_{LP}[n] = \begin{cases} 
    w[n]h_{LP,i}[n - (M - 0.5)] & 0 \leq n \leq (2M - 1) \\
    0 & \text{otherwise}
  \end{cases}
\]

where \( w[n] \) is a Hamming window defined for the samples \( 0 \leq m \leq 2M - 1 \):

\[
  w[n] = 0.54 - 0.46 \cos \left( \frac{2\pi n}{2M - 1} \right)
\]
Even Length Filters using Delay and Windowing

- **Truncated Delayed $l[n]$, cutoff=$\pi/4$**
  - Line graph showing the truncated delayed signal.

- **Hamming Window $w[n]$, Length=20**
  - Line graph showing the Hamming window.

- **Windowed Delayed Filter $w[n]h_{LP}[n - 9.5]$, Length=21**
  - Line graph showing the windowed delayed filter.
Even Length Filters using Delay and Windowing

$h_{LP,i}(n)$, cutoff $= \pi/4$

$H_{LP,i}(\omega)$, cutoff $= \pi/4$

Windowed $h_{LP}[n]$, cutoff $= \pi/4$

Windowed $|H_{LP}(\omega)|$, cutoff $= \pi/4$
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Summary: Ideal Filters

- **Ideal Lowpass Filter:**
  \[
  H_{LP}(\omega) = \begin{cases} 
  1 & |\omega| \leq \omega_c, \\
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  \end{cases}
  \quad \leftrightarrow \quad h_{LP}[m] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n)
  \]

- **Ideal Highpass Filter:**
  \[
  H_{HP}(\omega) = 1 - H_{LP}(\omega) \quad \leftrightarrow \quad h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi} \text{sinc}(\omega_c n)
  \]

- **Ideal Bandpass Filter:**
  \[
  H_{BP}(\omega) = H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega)
  \leftrightarrow h_{BP}[n] = \frac{\omega_2}{\pi} \text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi} \text{sinc}(\omega_1 n)
  \]
Summary: Practical Filters

- Odd Length:

\[ h_{HP}[n] = \begin{cases} 
  h_{HP,i}[n]w[n] & -M \leq n \leq M \\
  0 & \text{otherwise}
\end{cases} \]

- Even Length:

\[ h_{HP}[n] = \begin{cases} 
  h_{HP,i}[n - (M - 0.5)]w[n] & 0 \leq n \leq 2M - 1 \\
  0 & \text{otherwise}
\end{cases} \]

where \( w[n] \) is a window with tapered ends, e.g.,

\[ w[n] = \begin{cases} 
  0.54 - 0.46 \cos \left( \frac{2\pi n}{L-1} \right) & 0 \leq n \leq L - 1 \\
  0 & \text{otherwise}
\end{cases} \]
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Design a bandpass filter with lower and upper cutoffs of $\omega_1 = \frac{\pi}{3}$, $\omega_2 = \frac{\pi}{2}$, and with a length of $N = 33$ samples, using a Hamming window.