Problem 3.1

In MP3, one of the filters you’ll create is a local averaging filter. A local averaging filter produces an output $y[n]$, at time $n$, which is the average of the previous $N$ samples of $x[m]$:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n - m] \quad \text{(3.1-1)}$$

$$h[m] = \begin{cases} \frac{1}{N} & 0 \leq m \leq N - 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{(3.1-2)}$$

(a) First, consider what happens if $x[m]$ is a pure tone with a period of $N_0 = \frac{2\pi}{\omega_0}$, an amplitude of $A$, and a phase of $\theta$:

$$x[n] = A \cos (\omega_0 n - \theta)$$

Suppose that the averaging window, $N$, is exactly an integer multiple of $N_0$. For example, suppose that $N = 3N_0$. Draw a picture of $x[n]$ as a function of $n$, and shade in the regions that would be added together by the summation in Eq. (3.1-1) in order to compute $y[0]$. Argue based on your figure (with no calculations at all) that $y[0] = 0$.

(b) Adding up the samples of a cosine is easy when $N$ is an integer multiple of $N_0$, but hard otherwise. It’s actually much easier to add the samples of a complex exponential, because we can use the standard geometric series formula (https://en.wikipedia.org/wiki/Geometric_series#Formula). Use that formula to find $y[N - 1]$ when

$$x[n] = e^{j\omega_0 n}$$

Your result should have the form $y[0] = (1 - a)/(1 - b)$ for some complex-valued constants, $a$ and $b$, that depend on $\pi$, $N$, and $\omega_0$, but not on $m$ or $n$.

Problem 3.2

Another of the filters you’ll create in MP3 is a backward-difference filter. A backward-difference filter is one of several different ways of approximating a first-derivative:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n - m] \quad \text{(3.2-1)}$$

$$h[m] = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{(3.2-2)}$$
(a) First, consider what happens if $x[m]$ is a pure tone with a period of $T_0 = \frac{2\pi}{\omega_0}$, an amplitude of $A$, and a phase of $\theta$:

$$x[n] = A \cos (\omega_0 n - \theta) \quad (3.2-3)$$

Plug Eq. (3.2-3) into Eq. (3.2-1), then use the following trigonometric identity and the following approximation to discover in exactly what way the first-difference operator approximates a derivative:

$$-2 \sin(a) \sin(b) = \cos(a + b) - \cos(a - b)$$

$$\sin(b) \approx b \text{ if } b \text{ is small}$$

In order to apply the approximation, you can assume that $\omega_0$ is a small number.

(b) Let’s try the same thing with a complex exponential. Plug the following value of $x[n]$ into Eq. (3.2-1)

$$x[n] = e^{j\omega_0 n},$$

then assume that $\omega_0$ is a small number, and simplify using the approximation

$$e^\phi \approx 1 + \phi \text{ if } \phi \text{ is small}$$

in order to get something that looks like $dx[n]/dn$. 