Lecture 12: Autoregressive Filters

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ECE 401: Signal and Image Analysis, Fall 2020

Outline

1. Review: Z Transform
2. Autoregressive Difference Equations
3. Finite vs. Infinite Impulse Response
4. Impulse Response and Transfer Function of a First-Order Autoregressive Filter
5. Finding the Poles and Zeros of H(z)
6. Summary

Summary: Z Transform

- A **difference equation** is an equation in terms of time-shifted copies of $x[n]$ and/or $y[n]$.
- We can find the frequency response $H(\omega) = Y(\omega)/X(\omega)$ by taking the DTFT of each term of the difference equation. This will result in a lot of terms of the form $e^{j\omega n_0}$ for various n_0.
- We have less to write if we use a new frequency variable, $z = e^{j\omega}$. This leads us to the Z transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Zeros of the Transfer Function

- The **transfer function**, $H(z)$, is a polynomial in z.
- The zeros of the transfer function are usually complex numbers, z_k.
- The frequency response, $H(\omega) = H(z)|_{z=e^{j\omega}}$, has a dip whenever ω equals the phase of any of the zeros, $\omega = \angle z_k$.

Outline

Autoregressive Difference Equations

An **autoregressive** filter is one in which the output, $y[n]$, depends on past values of itself (**auto**=self, **regress**=go back). For example,

$$y[n] = x[n] + 0.3x[n-1] + 0.8y[n-1]$$

Autoregressive Difference Equations

We can find the transfer function by taking the Z transform of each term in the equation:

$$y[n] = x[n] + 0.3x[n-1] + 0.8y[n-1]$$
$$Y(z) = X(z) + 0.3z^{-1}X(z) + 0.8z^{-1}Y(z)$$

Transfer Function

In order to find the transfer function, we need to solve for $H(z) = \frac{Y(z)}{X(z)}$.

$$\begin{aligned} Y(z) &= X(z) + 0.3z^{-1}X(z) + 0.8z^{-1}Y(z) \\ \left(1 - 0.8z^{-1}\right) Y(z) &= X(z)(1 + 0.3z^{-1}) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{1 + 0.3z^{-1}}{1 - 0.8z^{-1}} \end{aligned}$$

Frequency Response

As before, we can get the frequency response by just plugging in $z = e^{j\omega}$. Some autoregressive filters are unstable,[1] but if the filter is stable, then this works:

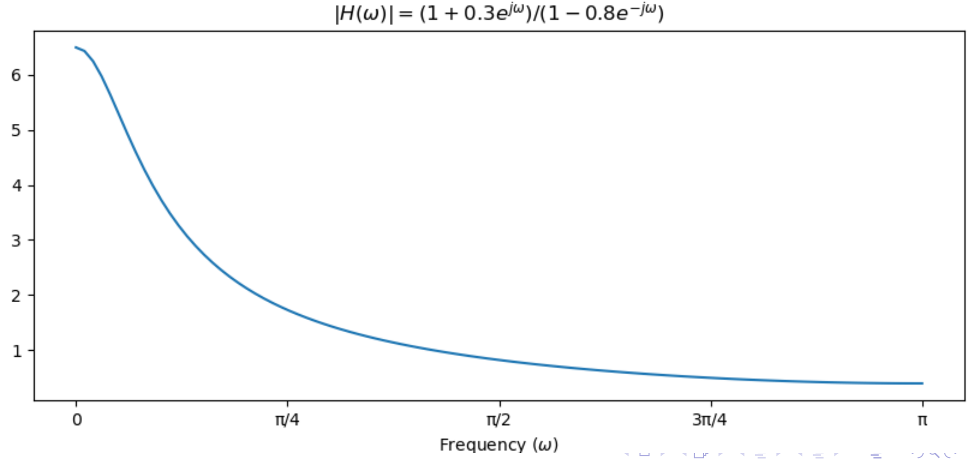
$$H(\omega) = H(z)\big|_{z=e^{j\omega}} = \frac{1 + 0.3e^{-j\omega}}{1 - 0.8e^{-j\omega}}$$

[1] "Unstable" means that the output can be infinite, even with a finite input. More about this later in the lecture.

Frequency Response

So, already we know how to compute the frequency response of an autoregressive filter. Here it is, plotted using

```
np.abs((1+0.3*np.exp(-1j*omega))/(1-0.8*np.exp(-1j*omega))
```



Outline

Impulse Response of an Autoregressive Filter

One way to find the **impulse response** of an autoregressive filter is the same as for any other filter: feed in an impulse, $x[n] = \delta[n]$, and what comes out is the impulse response, $y[n] = h[n]$.

$$h[n] = \delta[n] + 0.3\delta[n-1] + 0.8h[n-1]$$

$$\begin{aligned}
h[n] &= 0, \quad n < 0 \\
h[0] &= \delta[0] = 1 \\
h[1] &= 0 + 0.3\delta[0] + 0.8h[0] = 1.1 \\
h[2] &= 0 + 0 + 0.8h[1] = 0.88 \\
h[3] &= 0 + 0 + 0.8h[2] = 0.704 \\
&\vdots \\
h[n] &= 1.1(0.8)^{n-1} \quad \text{if } n \geq 1 \\
&\vdots
\end{aligned}$$

FIR vs. IIR Filters

- An autoregressive filter is also known as an **infinite impulse response (IIR)** filter, because $h[n]$ is infinitely long (never ends).
- A difference equation with only feedforward terms (like we saw in the last lecture) is called a **finite impulse response (FIR)** filter, because $h[n]$ has finite length.

General form of an FIR filter

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

This filter has an impulse response ($h[n]$) that is $M+1$ samples long.

- The b_k's are called **feedforward** coefficients, because they feed $x[n]$ forward into $y[n]$.

General form of an IIR filter

$$\sum_{\ell=0}^{N} a_\ell y[n-\ell] = \sum_{k=0}^{M} b_k x[n-k]$$

- The a_ℓ's are caled **feedback** coefficients, because they feed $y[n]$ back into itself.

General form of an IIR filter

$$\sum_{\ell=0}^{N} a_\ell y[n-\ell] = \sum_{k=0}^{M} b_k x[n-k]$$

Example:

$$\begin{aligned} y[n] &= x[n] + 0.3x[n-1] + 0.8y[n-1] \\ b_0 &= 1 \\ b_1 &= 0.3 \\ a_0 &= 1 \\ a_1 &= -0.8 \end{aligned}$$

Outline

First-Order Feedback-Only Filter

Let's find the general form of $h[n]$, for the simplest possible autoregressive filter: a filter with one feedback term, and no feedforward terms, like this:

$$y[n] = x[n] + ay[n-1],$$

where a is any constant (positive, negative, real, or complex).

Impulse Response of a First-Order Filter

We can find the impulse response by putting in $x[n] = \delta[n]$, and getting out $y[n] = h[n]$:

$$h[n] = \delta[n] + ah[n-1].$$

Recursive computation gives

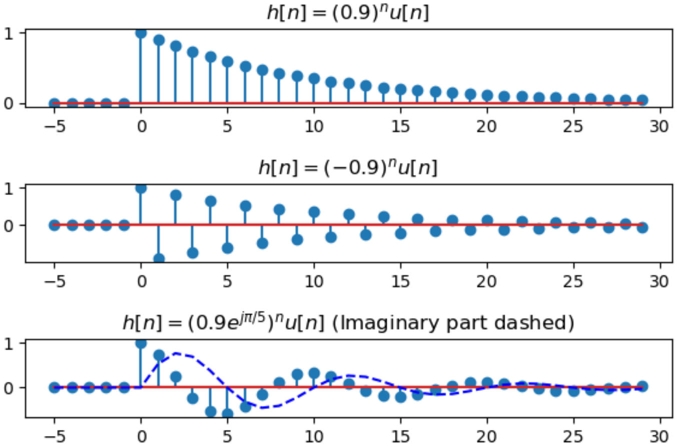
$$\begin{aligned} h[0] &= 1 \\ h[1] &= a \\ h[2] &= a^2 \\ &\vdots \\ h[n] &= a^n u[n] \end{aligned}$$

where we use the notation $u[n]$ to mean the "unit step function,"

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

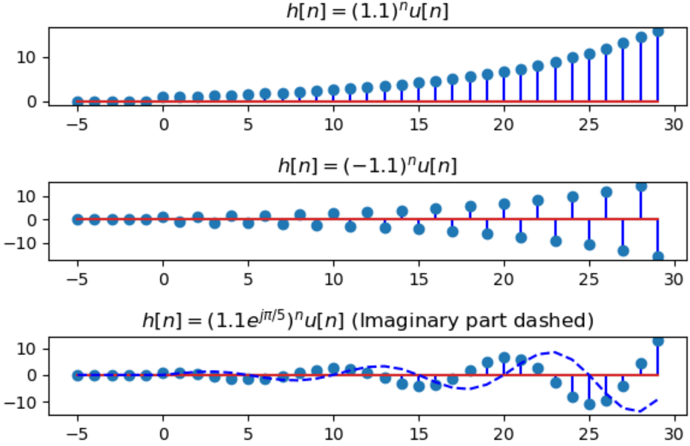
Impulse Response of Stable First-Order Filters

The coefficient, a, can be positive, negative, or even complex. If a is complex, then $h[n]$ is also complex-valued.



Impulse Response of Unstable First-Order Filters

If $|a| > 1$, then the impulse response grows exponentially. If $|a| = 1$, then the impulse response never dies away. In either case, we say the filter is "unstable."



Instability

- A **stable** filter is one that always generates finite outputs ($|y[n]|$ finite) for every possible finite input ($|x[n]|$ finite).
- An **unstable** filter is one that, at least sometimes, generates infinite outputs, even if the input is finite.
- A first-order IIR filter is stable if and only if $|a| < 1$.

Transfer Function of a First-Order Filter

We can find the transfer function by taking the Z-transform of each term in this equation equation:

$$y[n] = x[n] + ay[n-1],$$
$$Y(z) = X(z) + az^{-1}Y(z),$$

which we can solve to get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

Frequency Response of a First-Order Filter

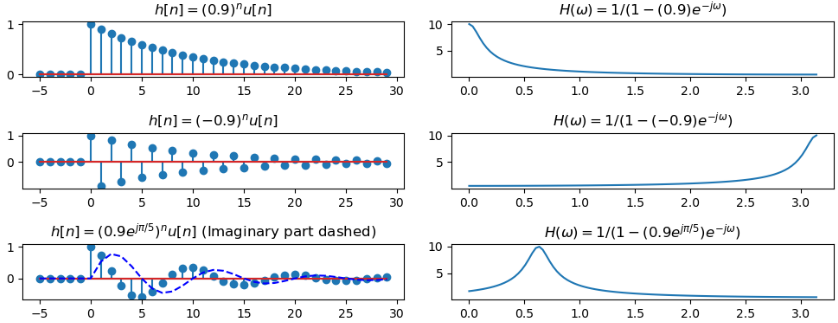
If the filter is stable ($|a| < 1$), then we can find the frequency response by plugging in $z = e^{j\omega}$:

$$H(\omega) = H(z)|_{z=e^{j\omega}} = \frac{1}{1 - ae^{-j\omega}} \quad \text{if } |a| < 1$$

This formula only works if $|a| < 1$.

Frequency Response of a First-Order Filter

$$H(\omega) = \frac{1}{1 - ae^{-j\omega}} \quad \text{if } |a| < 1$$



Transfer Function ↔ Impulse Response

For FIR filters, we say that $h[n] \leftrightarrow H(z)$ are a Z-transform pair. Let's assume that the same thing is true for IIR filters, and see if it works.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$
$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

This is a standard geometric series, with a ratio of az^{-1}. As long as $|a| < 1$, we can use the formula for an infinite-length geometric series, which is:

$$H(z) = \frac{1}{1 - az^{-1}},$$

So we confirm that $h[n] \leftrightarrow H(z)$ for both FIR and IIR filters, as long as $|a| < 1$.

Outline

First-Order Filter

Now, let's find the transfer function of a general first-order filter, including BOTH feedforward and feedback delays:

$$y[n] = x[n] + bx[n-1] + ay[n-1],$$

where we'll assume that $|a| < 1$, so the filter is stable.

Transfer Function of a First-Order Filter

We can find the transfer function by taking the Z-transform of each term in this equation equation:

$$y[n] = x[n] + bx[n-1] + ay[n-1],$$
$$Y(z) = X(z) + bz^{-1}X(z) + az^{-1}Y(z),$$

which we can solve to get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + bz^{-1}}{1 - az^{-1}}.$$

Treating $H(z)$ as a Ratio of Two Polynomials

Notice that $H(z)$ is the ratio of two polynomials:

$$H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}} = \frac{z + b}{z - a}$$

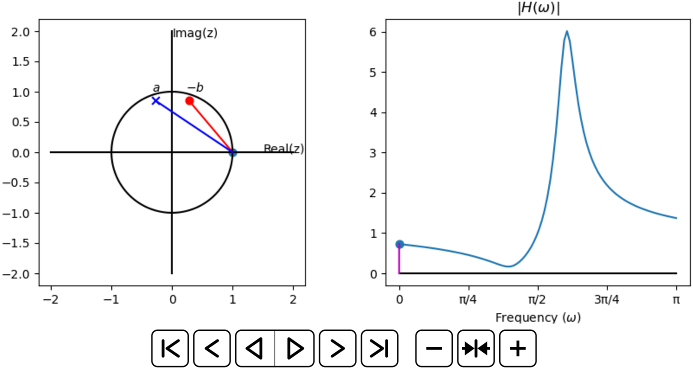
- $z = -b$ is called the **zero** of $H(z)$, meaning that $H(-b) = 0$.
- $z = a$ is called the **pole** of $H(z)$, meaning that $H(a) = \infty$

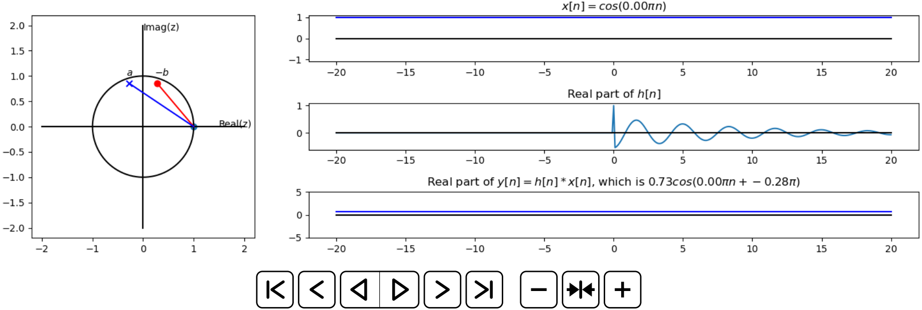
The Pole and Zero of $H(z)$

- The pole, $z = a$, and zero, $z = -b$, are the values of z for which $H(z) = \infty$ and $H(z) = 0$, respectively.
- But what does that mean? We know that for $z = e^{j\omega}$, $H(z)$ is just the frequency response:

$$H(\omega) = H(z)\big|_{z=e^{j\omega}}$$

  but the pole and zero do not normally have unit magnitude.
- What it means is that:
  - When $\omega = \angle(-b)$, then $|H(\omega)|$ is as close to a zero as it can possibly get, so at that that frequency, $|H(\omega)|$ is as low as it can get.
  - When $\omega = \angle a$, then $|H(\omega)|$ is as close to a pole as it can possibly get, so at that that frequency, $|H(\omega)|$ is as high as it can get.


Imag(z)
Real(z)
a
−b
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
|H(ω)|
6
5
4
3
2
1
0
0
π/4
π/2
3π/4
π
Frequency (ω)


Imag(z)
Real(z)
a
−b
$x[n] = cos(0.00\pi n)$
Real part of $h[n]$
Real part of $y[n] = h[n] * x[n]$, which is $0.73cos(0.00\pi n + -0.28\pi)$

Vectors in the Complex Plane

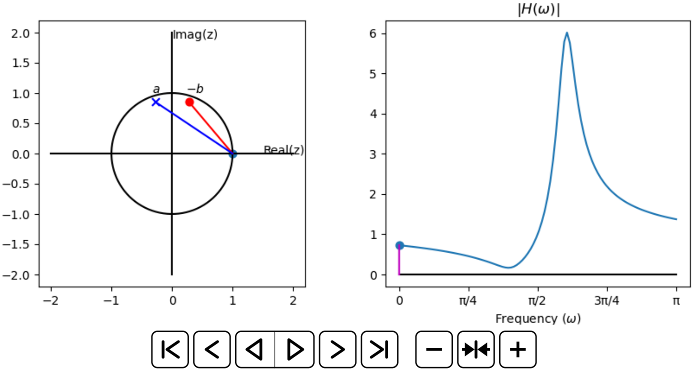
Suppose we write $|H(z)|$ like this:

$$|H(z)| = \frac{|z+b|}{|z-a|}$$

Now let's evaluate at $z = e^{j\omega}$:

$$|H(\omega)| = \frac{|e^{j\omega}+b|}{|e^{j\omega}-a|}$$

What we've discovered is that $|H(\omega)|$ is small when the vector distance $|e^{j\omega}+b|$ is small, but LARGE when the vector distance $|e^{j\omega}-a|$ is small.


Imag(z)
Real(z)
a
−b
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
|H(ω)|
6
5
4
3
2
1
0
0
π/4
π/2
3π/4
π
Frequency (ω)

Why This is Useful

Now we have another way of thinking about frequency response.

- Instead of just LPF, HPF, or BPF, we can design a filter to have zeros at particular frequencies, $\angle(-b)$, AND to have poles at particular frequencies, $\angle a$,
- The magnitude $|H(\omega)|$ is $|e^{j\omega}+b|/|e^{j\omega}-a|$.
- Using this trick, we can design filters that have much more subtle frequency responses than just an ideal LPF, BPF, or HPF.

Outline

Summary: Autoregressive Filter

- An **autoregressive filter** is a filter whose current output, $y[n]$, depends on past values of the output.
- An autoregressive filter is also called **infinite impulse response (IIR)**, because $h[n]$ has infinite length.
- A filter with only feedforward coefficients, and no feedback coefficients, is called **finite impulse response (FIR)**, because $h[n]$ has finite length (its length is just the number of feedforward terms in the difference equation).
- The first-order, feedback-only autoregressive filter has this impulse response and transfer function:

$$h[n] = a^n u[n] \leftrightarrow H(z) = \frac{1}{1 - az^{-1}}$$

Summary: Poles and Zeros

A first-order autoregressive filter,

$$y[n] = x[n] + bx[n-1] + ay[n-1],$$

has the impulse response and transfer function

$$h[n] = a^n u[n] + ba^{n-1}u[n-1] \leftrightarrow H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}},$$

where a is called the **pole** of the filter, and $-b$ is called its **zero**.