Lecture 5: Fourier Series and Discrete Fourier Transform

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ECE 401: Signal and Image Analysis, Fall 2020

Outline

Two-sided spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\left(x(t)\right) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Fourier's theorem

One reason the spectrum is useful is that **any** periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t+T_0) = x(t)$$

can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

which is a special case of the spectrum for periodic signals: $f_k = kF_0$, and $a_k = X_k$, and

$$F_0 = \frac{1}{T_0}$$

Analysis and Synthesis

- **Fourier Synthesis** is the process of generating the signal, $x(t)$, given its spectrum. Last lecture, you learned how to do this, in general.
- **Fourier Analysis** is the process of finding the spectrum, X_k, given the signal $x(t)$. I'll tell you how to do that today.

Outline

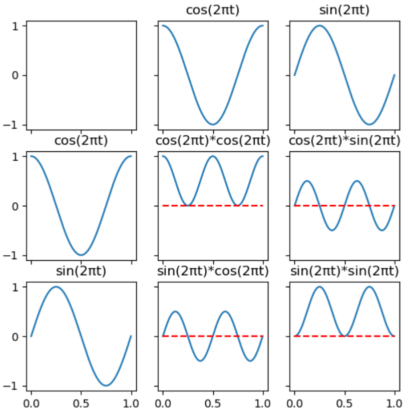
Orthogonality

Two functions $f(t)$ and $g(t)$ are said to be **orthogonal**, over some period of time T, if

$$\int_0^T f(t)g(t) = 0$$

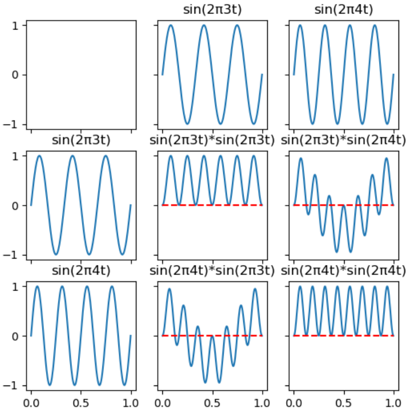
Sine and Cosine are Orthogonal

For example, $\sin(2\pi t)$ and $\cos(2\pi t)$ are orthogonal over the period $0 \leq t \leq 1$:



Sinusoids at Different Frequencies are Orthogonal

Similarly, sinusoids at different frequencies are orthogonal over any time segment that contains an integer number of periods:



How to use orthogonality

Suppose we have a signal that is known to be

$$x(t) = a\cos(2\pi 3t) + b\sin(2\pi 3t) + c\cos(2\pi 4t) + d\sin(2\pi 4t) + \ldots$$

...but we don't know a, b, c, d, etc. Let's use orthogonality to figure out the value of b:

$$\begin{aligned}\int_0^1 x(t)\sin(2\pi 3t)dt = a&\int_0^1 \cos(2\pi 3t)\sin(2\pi 3t)dt\\ + b&\int_0^1 \sin(2\pi 3t)\sin(2\pi 3t)dt\\ + c&\int_0^1 \cos(2\pi 4t)\sin(2\pi 3t)dt\\ + e&\int_0^1 \sin(2\pi 4t)\sin(2\pi 3t)dt + \ldots\end{aligned}$$

How to use orthogonality

. . . which simplifies to

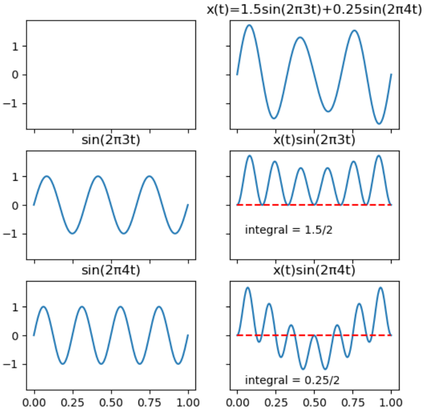
$$\int_0^1 x(t)\sin(2\pi 3t)dt = 0 + b\int_0^1 \sin^2(2\pi 3t)dt + 0 + 0 + \dots$$

The average value of $\sin^2(t)$ is $1/2$, so

$$\int_0^1 x(t)\sin(2\pi 3t)dt = \frac{b}{2}$$

If we **don't** know the value of b, but we **do** know how to integrate $\int x(t)\sin(2\pi 3t)dt$, then we can find the value of b from the formula above.

How to use orthogonality



How to use Orthogonality: Fourier Series

We still have one problem. Integrating $\int x(t)\cos(2\pi 4t)dt$ is hard—lots of ugly integration by parts and so on. There are two useful solutions, depending on the situation:

1. **Fourier Series:** Instead of cosine, use complex exponential:

$$\int x(t)e^{-j2\pi ft}dt$$

   That integral is still hard, but it's always easier than $\int x(t)\cos(2\pi 4t)dt$. It can usually be solved with some combination of integration by parts, variable substitution, etc.

2. **Discrete Fourier Transform:** Instead of integrating, write it as a sum:

$$\sum x[n]e^{-j2\pi fn/F_s}$$

   . . . and then write that as a line of python code, and solve it on the computer by typing `np.sum()`.

Outline

Fourier's Theorem

Remember Fourier's theorem. He said that any periodic signal, with a period of T_0 seconds, can be written

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

Fourier's Theorem and Orthogonality

Take Fourier's theorem, and multiply both sides by $e^{-j2\pi\ell t/T_0}$:

$$x(t)e^{-2\pi\ell t/T_0} = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi(k-\ell)t/T_0}$$

Now integrate both sides of that equation, over any complete period:

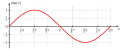
$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-2\pi\ell t/T_0}dt = \sum_{k=-\infty}^{\infty} X_k \frac{1}{T_0}\int_0^{T_0} e^{j2\pi(k-\ell)t/T_0}dt$$

Fourier's Theorem and Orthogonality

Now think really hard about what's inside that integral sign:

$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi(k-\ell)t/T_0}dt$$
$$=\frac{1}{T_0}\int_0^{T_0}\cos\left(\frac{2\pi(k-\ell)t}{T_0}\right)dt$$
$$+j\frac{1}{T_0}\int_0^{T_0}\sin\left(\frac{2\pi(k-\ell)t}{T_0}\right)dt$$

- If $k\neq\ell$, then we're integrating a cosine and a sine over $k-\ell$ periods. That integral is always zero.



- If $k=\ell$, then we're integrating

$$\frac{1}{T_0}\int_0^{T_0}\cos(0)dt++j\frac{1}{T_0}\int_0^{T_0}\sin(0)dt=1$$

Fourier Series: Analysis

So, because of orthogonality:

$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-2\pi\ell t/T_0}dt = \sum_{k=-\infty}^{\infty} X_k \frac{1}{T_0}\int_0^{T_0} e^{j2\pi(k-\ell)t/T_0}dt$$
$$= \ldots + 0 + 0 + 0 + X_\ell + 0 + 0 + 0 + \ldots$$

Fourier Series

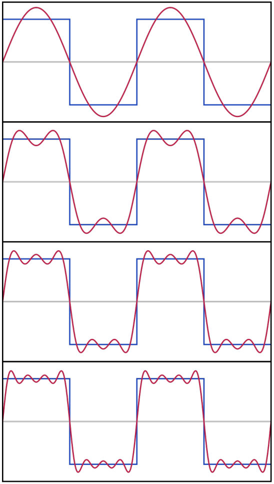
- **Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Synthesis** (finding the waveform, given the spectrum):

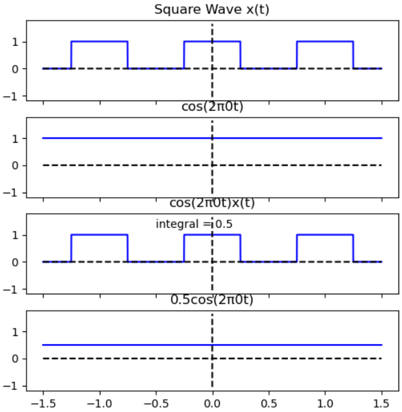
$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

Fourier series: Square wave example



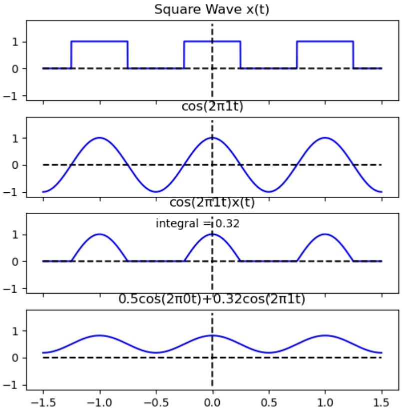
Square wave: the X_0 term

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)e^{j2\pi 0t/T_0}dt$$



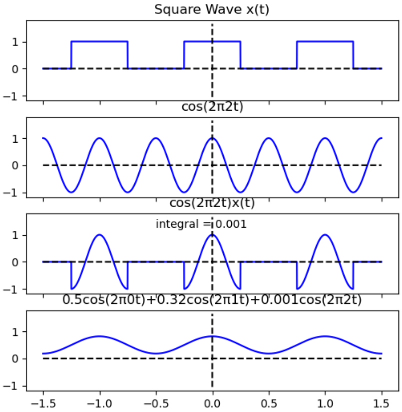
Square wave: the X_1 term

$$X_1 = \frac{1}{T_0}\int_0^{T_0} x(t)e^{j2\pi 1t/T_0}dt$$



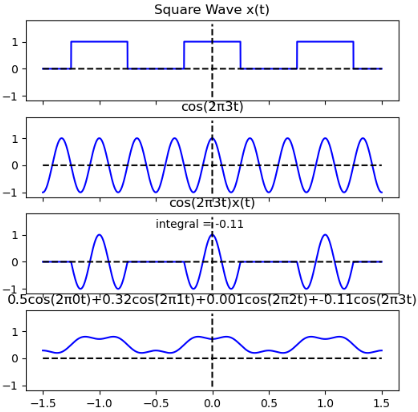
Square wave: the X_2 term

$$X_2 = \frac{1}{T_0}\int_0^{T_0} x(t)e^{j2\pi 2t/T_0}dt$$



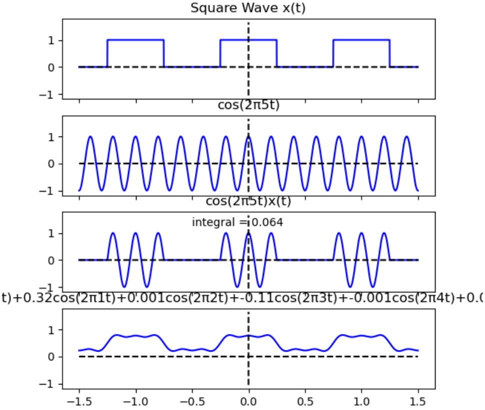
Square wave: the X_3 term

$$X_3 = \frac{1}{T_0}\int_0^{T_0} x(t)e^{j2\pi 3t/T_0}dt$$



Square wave: the X_5 term

$$X_5 = \frac{1}{T_0}\int_0^{T_0} x(t)e^{j2\pi 5t/T_0}dt$$



Fourier Series

- **Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

Outline

Discrete-time Fourier Series

Suppose you have a signal $x[n]$, sampled at a certain number of samples per second. Suppose $x[n]$ is periodic with a period of N samples, i.e.,

$$x[n] = x[n+N]$$

Then

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

Discrete Fourier Transform

Suppose you have a signal $x[n]$, sampled at a certain number of samples per second. We'll pretend that $x[n]$ is periodic with a period of N samples (even if it's not really), i.e., we'll pretend that

$$x[n] = x[n+N]$$

We'll define $X[k]$ so that

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$

. . . in other words, the DFT is just a scaled-up Fourier series.

Fourier's Theorem and Orthogonality

Take Fourier's theorem, and multiply both sides by $e^{-j2\pi\ell n/N}$:

$$x[n]e^{-2\pi\ell n/N} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi(k-\ell)n/N}$$

Now sum both sides of that equation:

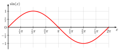
$$\sum_{n=0}^{N-1} x[n]e^{-2\pi\ell n/N} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\sum_{n=0}^{N-1} e^{j2\pi(k-\ell)n/N}$$

Fourier's Theorem and Orthogonality

Now think really hard about what's inside that summation:

$$\sum_{n=0}^{N-1} e^{j2\pi(k-\ell)n/N} dt$$
$$= \sum_{n=0}^{N-1} \cos\left(\frac{2\pi(k-\ell)n}{N}\right) + j\sum_{n=0}^{N-1} \sin\left(\frac{2\pi(k-\ell)n}{N}\right)$$

- If $k \neq \ell$, then we're summing a cosine and a sine over $k-\ell$ periods. That sum is always zero.



- If $k = \ell$, then we're summing

$$\sum_{n=0}^{N-1} \cos(0) + +j\sum_{n=0}^{N-1} \sin(0) = N$$

DFT: Analysis

So, because of orthogonality:

$$\sum_{n=0}^{N-1} x[n]e^{-2\pi\ell n/N} = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} e^{j2\pi(k-\ell)n/N}$$
$$= 0 + 0 + \ldots + 0 + 0 + X[\ell] + 0 + 0 + \ldots + 0 + 0$$

Discrete Fourier Transform

- **Analysis** (finding the spectrum, given the waveform):

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

- **Synthesis** (finding the waveform, given the spectrum):

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$

Outline

Summary

- **Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

- **DFT Analysis** (finding the spectrum, given the waveform):

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

- **DFT Synthesis** (finding the waveform, given the spectrum):

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}$$

Summary

Things you should know, from today's lecture:

- You don't need to memorize the equations on the previous page, but, for example, given the synthesis equation, you should be able to apply the orthogonality principle to derive the corresponding analysis equation.
- You should know that every periodic signal $x[n]$ has a spectrum $X[k]$, and that you can use `np.fft.fft` to compute it.
- You should know that the DFT makes an implicit assumption that the signal is periodic with a period of N samples. If the signal isn't really periodic, then you might get weird artifacts because of that assumption – minor at most time scales, but important for careful music analysis.