Lecture 1: Review of Calculus and Complex Numbers

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ECE 401: Signal and Image Analysis, Fall 2020

Outline

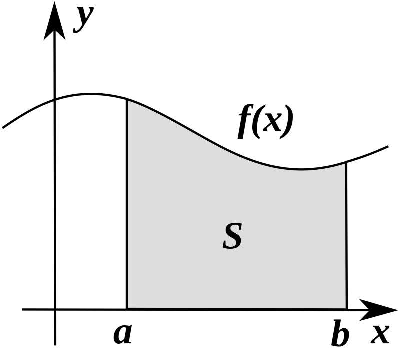
Outline of today's lecture

1. **Syllabus**
2. **Homework 1**
3. **Textbook**
4. Review: Integration, Summation, and Complex numbers

Outline

1. Outline of today's lecture
2. Review: How to integrate an exponential
3. Review: Summing a geometric series
4. Review: Complex numbers
5. Summary

Integration = Computing the area under a curve



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Why does signal processing use integrals?

- Real-world **signals** are functions of continuous time, or space, or both. For example, sound is air pressure as a function of time, $p(t)$.
- The **energy** necessary to produce a signal depends on its long-term integral:

$$E \propto \int_{-\infty}^{\infty} p^2(t)dt$$

- The **information** in a signal is encoded at different frequencies (f), which we can get using something called a **Fourier transform**. For continuous-time signals, a Fourier transform is an integral:

$$P(f) = \int_{-\infty}^{\infty} p(t)e^{-j2\pi ft}dt$$

Indefinite vs. definite integrals

- An **indefinite integral** (a.k.a. antiderivative) is the opposite of a derivative:

$$F(x) = \int f(x) \quad \text{means that} \quad f(x) = \frac{dF}{dx}$$

- A **definite integral** is the area under the curve. We can write it as:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Indefinite integrals worth knowing

- Integral of a polynomial:

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

- Integral of an exponential:

$$\int e^x dx = e^x$$

Methods for turning one integral into another

- Variable substitution: Suppose $f(x) = g(u)$ where u is some function of x. There is no other x anywhere inside $f(x)$. Then:
$$\int g(u)dx = \int \frac{1}{du/dx} g(u)du$$
- Integration by parts:
$$\int udv = uv - \int vdu$$

Example: How to integrate an exponential

What is $\int_{0.4}^{1.6} e^{j((x+y)t+\theta)}dt$?

1. Pull out the constants:

$$\int_{0.4}^{1.6} e^{j((x+y)t+\theta)}dt = e^{j\theta}\int_{0.4}^{1.6} e^{j(x+y)t}dt$$

2. Prepare for variable substitution:

$$u = j(x+y)t \quad \text{means that} \quad \frac{du}{dt} = j(x+y)$$

$$t \in [0.4, 1.6] \quad \text{means that} \quad u \in [j(x+y)0.4, j(x+y)1.6]$$

3. Variable substitution:

$$\int_{0.4}^{1.6} e^{j(x+y)t}dt = \int_{j(x+y)0.4}^{j(x+y)1.6} \frac{1}{j(x+y)}e^u du$$

Example: How to integrate an exponential

4. Pull out the constants again:

$$\int_{j(x+y)0.4}^{j(x+y)1.6} \frac{1}{j(x+y)} e^u du = \frac{1}{j(x+y)} \int_{j(x+y)0.4}^{j(x+y)1.6} e^u du$$

5. Integrate:

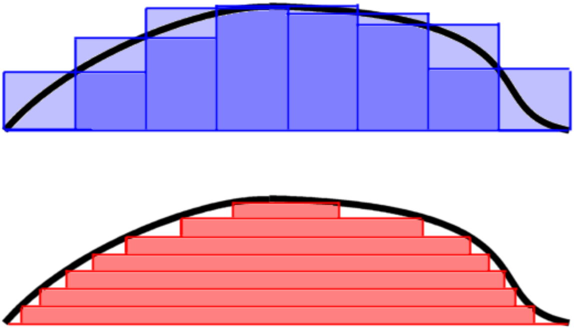
$$\int_{j(x+y)0.4}^{j(x+y)1.6} e^u du = [e^u]_{j(x+y)0.4}^{j(x+y)1.6}$$

6. Solve:

$$\int_{0.4}^{1.6} e^{j((x+y)t+\theta)} dt = \frac{e^{j\theta}}{j(x+y)} \left(e^{j(x+y)1.6} - e^{j(x+y)0.4} \right)$$

Outline

Summation is a computer-friendly version of integration



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Why does signal processing use sums?

- On computers, a signal is a sequence of numbers $x[n]$, regularly spaced samples of some real-world signal.
- The **energy** necessary to produce a signal depends on its long-term summation

$$E \propto \sum_{-\infty}^{\infty} x^2[n]$$

- The **information** in a signal is encoded at different frequencies (f), which we can get using something called a **Fourier transform**. For discrete-time signals, a Fourier transform is a summation

$$X(\omega) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$$

Sums worth knowing

- Exponential series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$$

- Geometric series:

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

Example: How to sum a geometric series

Problem: What is $\sum_{n=-7}^{7} e^{-j\omega n}$?

1. Prepare for variable substitution #1:

$$m = n + 7 \quad \text{means that} \quad n = m - 7$$
$$n \in [-7, 7] \quad \text{means that} \quad m \in [0, 14]$$

2. Variable substitution #1:

$$\sum_{n=-7}^{7} e^{-j\omega n} = \sum_{m=0}^{14} e^{-j\omega(m-7)}$$

3. Pull out the constants:

$$\sum_{m=0}^{14} e^{-j\omega(m-7)} = e^{7j\omega} \sum_{m=0}^{14} e^{-j\omega m}$$

Example: How to sum a geometric series

4. Prepare for variable substitution #2:

$$r = e^{-j\omega} \quad \text{means that} \quad e^{-j\omega m} = r^m$$

5. Variable substitution #2:

$$e^{7j\omega} \sum_{m=0}^{14} e^{-j\omega m} = e^{7j\omega} \sum_{m=0}^{14} r^m$$

6. Sum:

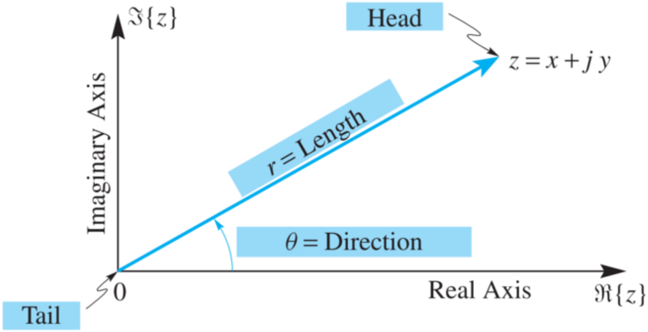
$$\sum_{m=0}^{14} r^m = \frac{1 - r^{15}}{1 - r}$$

7. Solve:

$$\sum_{n=-7}^{7} e^{-j\omega n} = e^{7j\omega} \left(\frac{1 - e^{-j15\omega}}{1 - e^{-j\omega}} \right)$$

Outline

Complex numbers



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Why does signal processing use complex numbers?

- The Fourier transform was originally defined in terms of cosines and sines:

$$X(\omega,\theta)=\int_{-\infty}^{\infty} x(t)\cos(\omega t+\theta)dt$$

- . . . but exponentials are easier to integrate than cosines, and **a lot** easier to sum.
- . . . so we take advantage of Euler's equation, to turn all of the cosines and sines into exponentials:

$$e^{j\omega t}=\cos(\omega t)+j\sin(\omega t)$$

Rectangular and polar coordinates

$$z = x + jy = me^{j\theta}$$

- Converting rectangular to polar coordinates:

$$m = \sqrt{x^2 + y^2}, \quad \theta = \begin{cases} \text{atan}\left(\frac{y}{x}\right) & x > 0 \\ \pm\frac{\pi}{2} & x = 0 \\ \text{atan}\left(\frac{y}{x}\right) \pm \pi & x < 0 \end{cases}$$

- Converting polar to rectangular:

$$x = m\cos\theta, \quad y = m\sin\theta$$

Outline

Summary

1. Integration:

$$\int e^x dx = e^x, \qquad \int g(u) dx = \int \frac{1}{du/dx} g(u) du$$

2. Summation

$$\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$$

3. Complex numbers:

$$m = \sqrt{x^2+y^2}, \quad \theta = \begin{cases} \operatorname{atan}\left(\frac{y}{x}\right) & x > 0 \\ \pm\frac{\pi}{2} & x = 0 \\ \operatorname{atan}\left(\frac{y}{x}\right) \pm \pi & x < 0 \end{cases}$$