# Agenda

Final Remarks about the course

- Where can I apply probabilities?
- Correlated advanced courses
- Probability mindset

Final Review I (Pre-midterm2)

- Conditional probability
- Functions of RV
- Distributions
- Combinations and Permutations

Next week

- Joint RV, MMSE

# Final Remarks

# Where can I apply probability?

- ML – VAE, Diffusion, Bayesian DL, RL
  - Law of Large Number is everywhere in data. Noise is inherent

- Communication networks – Queueing, error coding, fading
  - Every bits is a Bernoulli

- Signal Processing – Kalman filter, Wiener filer, spectral analysis
  - Noise is the signal’s shadow — learn the shadow, and you master the signal
- Robotics & control – Tracking, SLAM, sensor fusion
  - Model the unknowns – That’s reality

# Where can I apply probability?

- Computer Security – Cryptography strength, side channel attacks
  - Security is fundamentally a probability game

- Semiconductor Devices – Reliability & lifetime analysis
  - At microscopic scales, randomness is the rule, not the exception

# What advanced courses are correlated

- ECE 418 — Image & Video Processing
- ECE 420 / 551 — DSP and Advanced DSP
- ECE 438 — Communication Networks
- ECE 434 — Real-World Algorithms for IoT and Data Science
- ECE 448/449/494 — AI/ ML
- ECE 486 / 489 / 515 — Control Systems
- ECE 498/598RR — Deep Generative Models
- ECE 534 — Random Process
- ECE 543 — Statistical Learning Theory
- ECE 561 — Statistical Inference for Engineers and Data Scientists
- And more...

# Probability mindset

- What is the value? → What is the distribution?
  - Always assume a RV, collapse to constant when $\sigma \approx 0$
- Uncertainty is a chance, not a nuisance
  - Model Gaussian, and focus on None-Gaussian
- What distribution would generate observation $X$?
  - Diffusion, Kalman, Markov model, Likelihood
- When in doubt: model uncertainty explicitly
  - What varies? How does it affect the output

# Closing Messages

- Probability isn’t a chapter in the textbook—it’s the operating system of the real world.
- If there is one habit to carry forward from ECE 313, let it be: Whenever something looks uncertain, irregular, or ‘noisy’—don’t guess, don’t simplify—model it.
- Great engineers don’t eliminate randomness; they understand it, quantify it, and use it.

# Final Review I (Pre-midterm2)

# Conditional Probability

$$P(B|A) = \begin{cases} \frac{P(AB)}{P(A)} & if\ P(A) > 0 \\ undefined & else \end{cases}$$

Toss a coin two times, denote the result as $X_1$ and $X_2$

- $P\{X_2 = H | X_1 = H\}$ $= \frac{1}{2}$ $X_2$ ind. $X_1$
- $P(Exactly\ 1\ H\ in\ 2\ tosses\ | X_1 = H\}$

$\frac{P(\{HH\})}{P(\{HX\})} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$ $\to$ $\frac{P\{HL\}}{P\{HX\}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

$C, H_1, H_2$ → pairwise independent?

$P\{C \mid H_1, H_2\} \neq P\{C\}$ ⇒ $C, H_1, H_2$ not independent

$P\{C \mid H_1\} = P(C)$ & $P\{C \mid H_2\} = P(C)$

& $P\{H_1 \mid H_2\} = P(H_1)$

# Conditional Probability

- For RV $X$ and $Y$, if $P(X|Y) = P(X)$, $X$ and $Y$ are independent
- For jointly distributed $X$ and $Y$, $f_{X|Y}(u|v) = \frac{f_{XY}(u,v)}{f_Y(v)}$
  - A “normalized slice” of joint PDF

    ↳ $Y = v$.

- Write the equations
  - Differentiate between $P(Y)$, $P(X|Y)$, $P(Y|X)$, $P(XY)$

- BHT
  - $p_{miss} = P\{Claim\ H_0|H_1\}$

# Examples

- A Bag contains two biased coins $a, b$, $[p_a, p_b] = [.1, .5]$
  - Draw a coin and toss it 1000 times, get 600 $H$, what's the probability I draw $a$
  - Draw a coin and toss, get $H$, and toss it again, $P(H) = ?$

$P\{X=a|T\} = \frac{P(T|X=a)\,P(X=a)}{P(T)}$, $P(X=a) = \frac{1}{2}$

$P(T) = P(T|X=a)P(X=a) + P(T|X=b)P(X=b)$

optional

$H_a = N_a \sim (100, 1000 \times 0.1 \times 0.9 = 90)$

$599.5 \le H_a \le 600.5$

$$P(X=a|H_1) = \frac{P(H_1|a)\,P(a)}{P(H_1)} = \frac{0.1 \times 0.5}{0.1\times 0.5 + 0.5 \times 0.5} = \frac{1}{6}$$

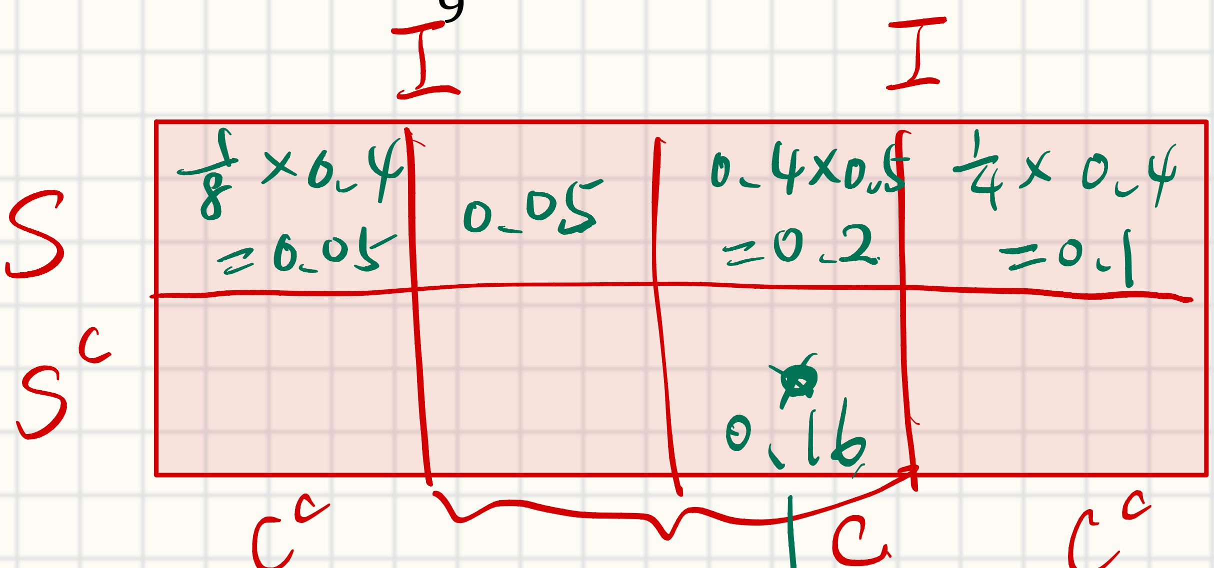
$$\frac{1}{6} \times P(H_2|a,H_1) + \frac{5}{6} \times P(H_2|b,H_1)$$

$\frac{1}{6} \downarrow P(a|H_1)$

$$= \frac{1}{60} + \frac{25}{60} = \frac{13}{30}$$

# Examples

- Alice works in a coffee shop, the drink has 3 attributes
  - $I$ : ice or not, $S$: small or not, $C$: coffee or not
  - $P(S) = 0.4, P(C) = 0.6$
  - $P(HC|S) = .5, P(HC^c|S) = .25, P(H^cC^c|S) = .125$
  - $I$ is independent of $C$
  - $P(S^c|HC) = \frac{4}{9}$



# Examples – Poisson Process

- Let $N_t$ denotes a Poisson process of rate $\lambda$
  - $P\{N_8 = 5 | N_5 - N_4 = 3\}$
  - $P\{N_5 - N_3 = 3 | N_4 - N_2 = 1\}$

# Functions of a random variable

# Find CDF/ PDF of g(X)

Motivation – I know $X$ follows some distribution

- but what about $Y = g(X)$?

1. Scope the problem - Find support of $X$ and $Y$, are they continuous or discrete?
2. Find $F_Y(c)$ from integrating $f_X(x)$ over $\{x: g(x) \leq c\}$
   - If $Y$ is discrete, normally we can find pmf $p_Y(c)$
3. Get $f_Y = F_Y{}'$

# Examples - General

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\leq c} f_X(x)dx$
3. $f_Y = F_Y'$

$X \sim Uniform$, $E[X] = 1, \sigma_X^2 = 3, Y = |X|$. Find $f_Y(v)$

$Z \sim Uniform[0,1]$ $\mu_Z = 0.5$ $\sigma_Z^2 = \frac{1}{12}$

$\Rightarrow X \sim Uniform[1-3, 1+3] = Uniform[-2, 4]$

$f_X(u) \begin{cases} \frac{1}{6} & u \in [\ ] \\ 0 & else \end{cases}$

$F_Y(v) = P\{|X| \leq v\} = P\{-v \leq X \leq v\}$

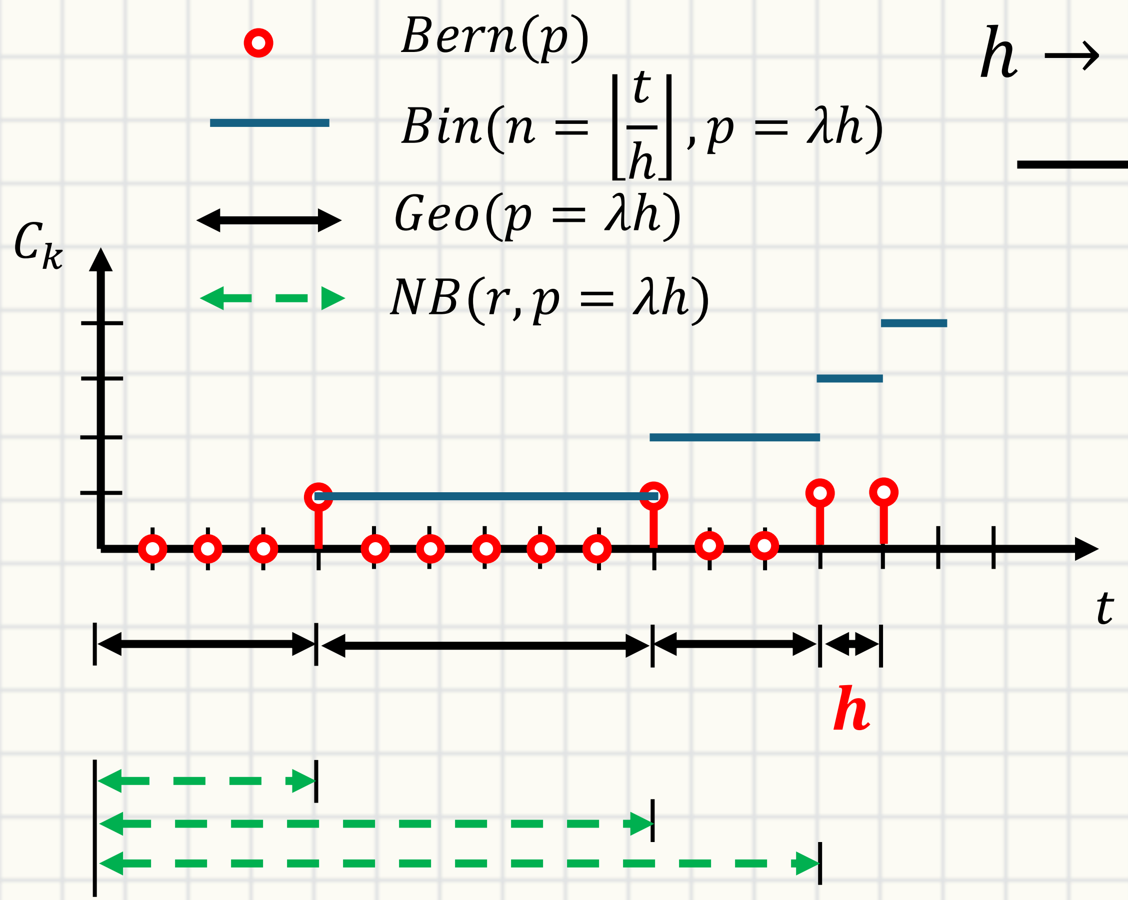
$$f_Y(v) = \begin{cases} \frac{1}{3}, & 0 < v \leq 2 \\ \frac{1}{6} & 2 < v \leq 4 \\ 0 & else \end{cases}$$

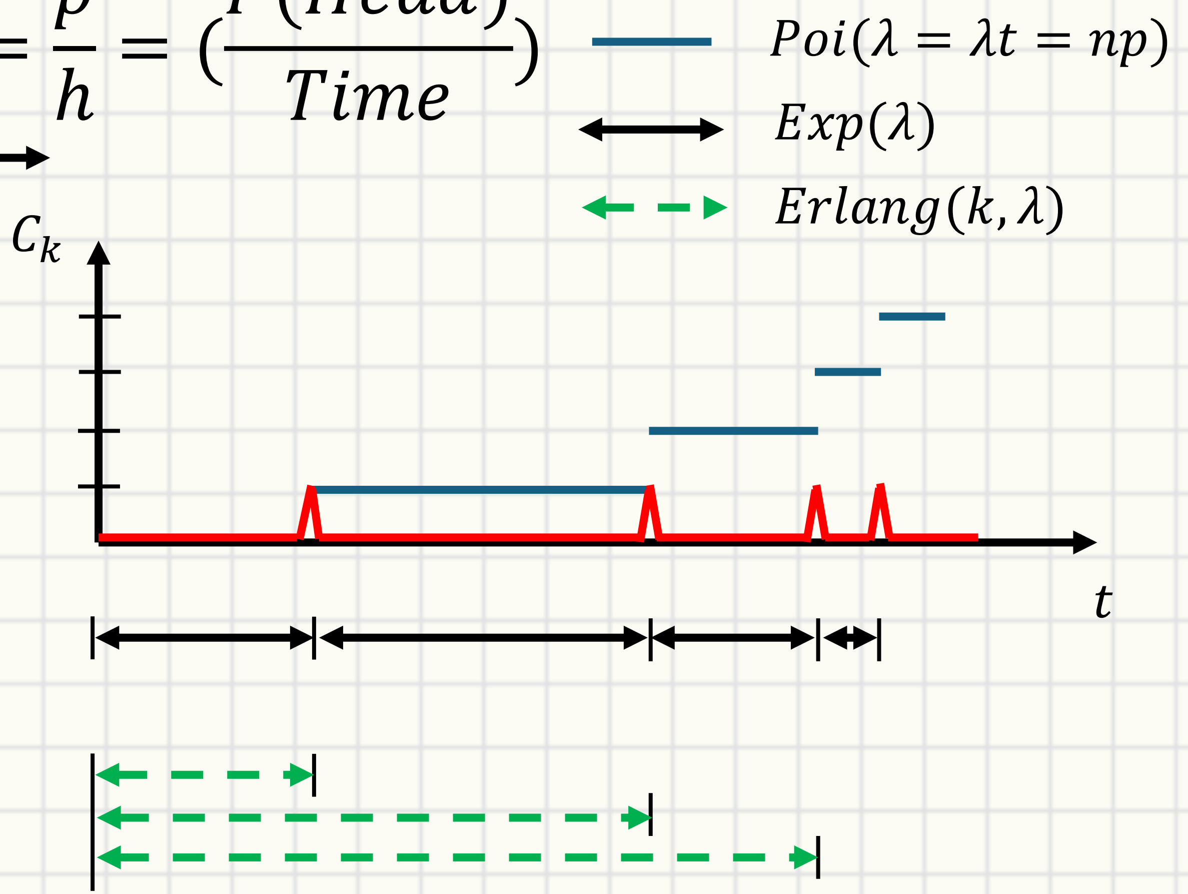
# Distributions

# Bernoulli Process $\xrightarrow{h \to 0, \lambda = \frac{p}{h}}$ Poisson Process

- Assume each trial takes $h$ duration to complete

$$h \to 0, \lambda = \frac{p}{h} = \left(\frac{P(Head)}{Time}\right)$$



# Properties

| | $Bern(p)$ | $Bin(n,p)$ | $Poi(\lambda = np)$ | $Geo(p)$ | $NB(r,p)$ |
|---|---|---|---|---|---|
| Def | $X_i$ | $\sum_n X_i$ | $\sum_n X_i, n \to \infty$ | $Y_i$ | $\sum_n Y_i$ |
| $\mu$ | $p$ | $np$ | $\lambda$ | $1/p$ | $r/p$ |
| $\sigma^2$ | $p(1-p)$ | $np(1-p)$ | $\lambda$ | $\frac{1-p}{p^2}$ | $\frac{r(1-p)}{p^2}$ |
| $f_X/p_X(k)$ | $p$ or $1-p$ | $\binom{n}{k} p^k (1-p)^{n-k}$ | $\frac{\lambda^k e^{-\lambda}}{k!}$ | $(1-p)^{k-1}p$ | $\binom{k-1}{r-1}(1-p)^{k-r}p^r$ |
| Special | | $(p+q)^n$ | $Poi(np) \approx Bin(n,p)$ | Memoryless | |

# Properties

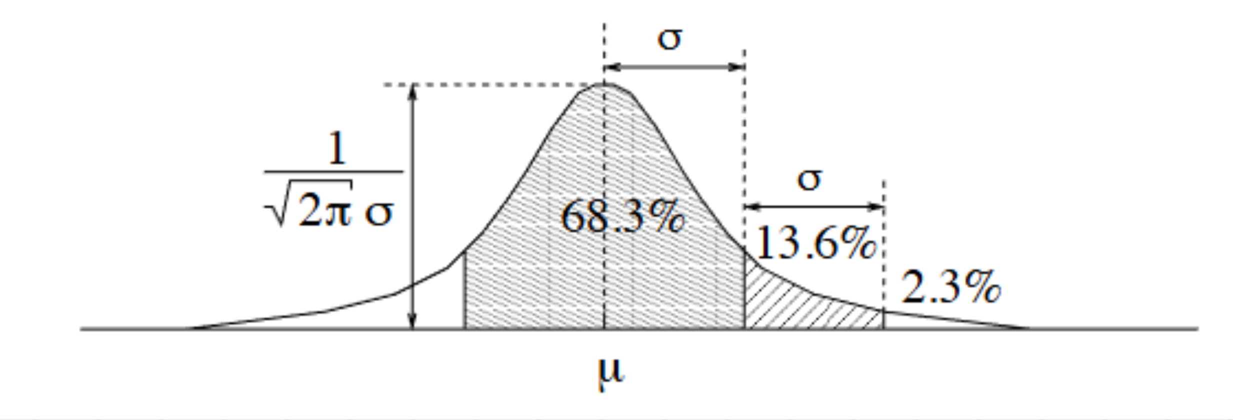
| | $Exp(\lambda)$ | $Poi(\lambda = \lambda t)$ | $Uni[a, b]$ | $N(\mu, \sigma^2)$ |
|---|---|---|---|---|
| Def | $T_i$ | $\sum_n X_i, n \to \infty$ | | |
| $\mu$ | $1/\lambda$ | $\lambda$ | $\frac{a+b}{2}$ | $\mu$ |
| $\sigma^2$ | $1/\lambda^2$ | $\lambda$ | $\frac{(b-a)^2}{12}$ | $\sigma^2$ |
| $f_X/p_X(k)$ | $\lambda e^{-\lambda t}$ | $\frac{\lambda^k e^{-\lambda}}{k!}$ | $\frac{1}{b-a}$ | $\frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$ |
| Special | Memoryless | | | $F_N(x) \triangleq \Phi(x)$ |

# Gaussian – Standard Normal

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$Z \sim N(0,1)$

- $\Phi(u) \triangleq F_Z(u) = \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})$
- $Q(u) = 1 - \Phi(u)$



# Gaussian – General Gaussian

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$X \sim N(\mu, \sigma^2)$

- $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

# Permutation/ Combination

# Permutation

- The # of ways to order n different items
- How many ways can you order letters $A, B, C, D$? $4!$

- What if I want to order "$A, B, C$ … $G$ " 7 letters, but only pick the first 4? $7 \times 6 \times 5 \times 4 = \frac{7!}{(7-4)!}$

- What if I want to order letters ILLINI?

$$\frac{6!}{3!\ 2!}$$

↑ Is ↑ Ls

# Combination

- $\binom{n}{k}$ or $C(n, k)$
  - The # of combination to choose k out of n different items
  - Doesn't care the order of they being picked
  - $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$
  - Draw 3 balls out of 5 balls without replacement
  - In most of cases, use combination on both nominator and denominator. -> unless order matters

# Binary Hypothesis Testing

- Likelihood matrix/ function
  - $f_i(k) = P\{X = k|H_i\}$
  - Likelihood Ratio $\Lambda(k) = \frac{f_1(k)}{f_0(k)}$
  - ML method - $f_1(k)\ vs.\ f_0(k)$ or $\Lambda(k) > 1$?

- Joint Probability Matrix $P(X_i H_i)$
  - MAP method - $\Lambda(k) > \tau_{MAP} = \frac{\pi_0}{\pi_1}$

# Example -

Let $f_1 \sim N(\mu = 1, \sigma^2 = 2)$ and $f_0 \sim N(\mu = 0, \sigma^2 = 1)$

- Find ML rule and MAP rule corresponding $P_{miss}, P_{false\ alarm}$

# Tuesday Review Session

- Joint Probability
- Estimators
- Functions of joint RV